

The Optimum Quantity of Money with Borrowing Constraints*

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July 26, 2011

Abstract

We characterize the optimal anticipated monetary policy in economies where agents have a precautionary savings motive due to random production opportunities and the presence of borrowing constraints. Non storable production makes intrinsically useless outside money valuable to insure consumption. We show that the choice of the optimal money growth rate trades off insurance vs. incentives to produce: an expansionary policy provides liquidity to borrowing constrained agents, but distorts production incentives. The joint presence of uncertainty and borrowing constraints implies that the Friedman rule leads to autarkic allocations. If the utility function satisfies Inada conditions then the optimal money growth rate is strictly positive and finite.

JEL classification: E50.

Keywords: Incomplete markets, Friedman rule, Optimal policy .

*We thank Fernando Alvarez, Paola Caselli, Isabel Correia, Piero Gottardi, Hugo Hopenhayn, David Levine, John Moore, Ezra Oberfield, Facundo Piguillem, Rob Shimer and Aleh Tsyvinski for useful discussions. We are grateful to seminar participants at EIEF's summer macro lunch seminars, EIEF's Macro Reading Group, the Bank of Italy, and the University of Edinburgh.

1 Introduction

We evaluate the optimal monetary policy in an economy where agents face random oscillations in their production opportunities which fluctuate between productive and unproductive periods. The economy is populated by agents of two types, defining their productive state. Types are assumed to be perfectly negatively correlated, so that at each point in time only one type is productive. As in the seminal work of [Scheinkman and Weiss \(1986\)](#) agents face a borrowing constraint: since the state is not observable agents cannot issue private debt. Because of this market incompleteness, money (an intrinsically useless object) may serve a fundamental insurance role and be valued. We show that the optimal policy is expansionary at a finite rate.

We extend [Scheinkman and Weiss's](#) analysis, which assumes a constant money supply, by letting the government choose a perfectly anticipated monetary growth rate, implemented through lump-sum money transfers. As in [Levine \(1991\)](#) we assume that the government does not know which agent is productive, so that the transfers are equal across agents.¹ In this economy money is the only savings instrument: an unproductive agent consumes exchanging money for goods with the productive agent. As the state can be reversed, the value of money is positive for productive agents too, who are hence willing to trade. A known feature of the [Scheinkman-Weiss](#) economy is that rich productive agents will be relatively less interested in trading goods for money. It follows that trade volumes, aggregate production, and the price of money depend on the distribution of wealth (i.e. shares of money holdings) which evolves through time following the history of shocks.

We provide an analytical characterization of the price of money and aggregate production, as functions of the money growth rate parameter and the wealth distribution. Moreover, we characterize the dynamics of the wealth distribution as a function of money growth and the history of shocks, as well as the invariant distribution of wealth. These objects give a

¹See [Kehoe, Levine and Woodford \(1990\)](#) for a thorough discussion of this assumption and in particular [Levine \(1991\)](#) for a careful derivation of the equal-treatment restriction from first principles.

complete analytical description of the dynamics of this economy.

Since the money growth rate affects the distribution of wealth, monetary policy has real effects and the choice of the optimal anticipated policy involves a tradeoff between two margins: the first one is that a monetary expansion provides insurance to agents who incur in a long spell of unproductive periods, and end-up having low money holdings and little consumption. The second margin is the classic cost of inflation: an expansionary policy lowers the return on money, lowering productive agents' incentives to save and produce. The optimal money growth rate trades off insurance vs. production incentives.

Related literature

A few previous contributions discuss environments where a flat monetary expansion is efficient in an economy with incomplete markets and where money serves an essential role (an extensive review appears in [Appendix A](#)). [Levine's \(1991\)](#) seminal paper considers an endowment economy where the agents' (bounded) utility functions change randomly according to whether they are "buyers" or "sellers", a state that follows an exogenous Markov process. In Levine's model sellers sell their entire endowment, which amounts to a restriction on the agents' marginal utilities over the set of feasible trades. Because of this assumption monetary policy can provide insurance at no cost since altering the relative price has no effect on welfare in the corner solution. [Kehoe, Levine and Woodford \(1990\)](#) extend Levine's setting to allow for internal solutions where sellers do not necessarily sell all their endowment. For reasons of tractability, they restrict attention to equilibria in which what happens in each period is *independent of history* (two-state markov equilibria). Compared to these papers, our contribution is to analyze the question of the optimal policy in the context of a production economy, and to focus on equilibria in which the decisions in each period depend on the whole history of shocks, as summarized by the distribution of money holdings.

In a broader sense the mechanism that we study is related to a recent strand of literature dealing with the interplay between the agents' liquidity and the business cycle, these include

Kiyotaki and Moore (2008), Guerrieri and Lorenzoni (2009), Chamley (2010), Brunnermeier and Sannikov (2011). Our results are also related to Aiyagari (1995) who studies a neoclassical growth model with borrowing constraints, as in Bewley (1977). Aiyagari shows that a positive capital tax rate is optimal because agents are over-saving due to precautionary motives and therefore the capital stock is too high. Because the government faces no incomplete markets, an optimal policy involves taxing capital and using the proceedings to produce a public good that all consumers enjoy, independently of their idiosyncratic shock. Our finding that the optimal policy involves an expansionary policy is reminiscent of Aiyagari’s result that the return on savings should be taxed. However the mechanism underlying these results is different: in Aiyagari’s it is assumed that the government observes the distribution of capital (or observes the investments made by the agents) and therefore is able to tax capital and use the proceeds to provide insurance to the agents, while in our model the government does not observe transactions nor the distribution of wealth and therefore is not able to tax agents.² In other words, Aiyagari’s government has higher powers and therefore can resort to a wider set of policies. Another difference with Aiyagari is that the economy we consider has cycles: aggregate production, the real interest rate, and the value of money fluctuate through time. Because of these features, the method that we use to characterize the optimal policy is different.

Overview of the analysis

Section 2 defines the economic environment: the agent’s utility functions, production possibilities, and the monetary transfer scheme. With fluctuating productive opportunities, the agents face an insurance problem. It is shown that without uncertainty the Townsend (1980), Bewley (1980) result on the optimality of the “Friedman rule” holds: agents are fully insured and the efficient complete markets allocation can be sustained. We then introduce uncertainty in production opportunities, and explore what the government can do using a

²We discuss in Section C how the government might actually support first best through taxation if it had the ability to commit to trigger policies. That is, we vary the power of the government

non-monetary instrument (direct taxation) under various environments. Assuming the government does not know which agent is productive leaves no role for direct taxation. In the same environment, the government can instead effectively use monetary policy. In a way, monetary policy can do something that direct taxation cannot do.

[Section 3](#) defines a monetary equilibrium, and the agent's optimality conditions, in the stochastic environment. In a similar fashion to [Bewley \(1983\)](#), it is shown in [Section 3.1](#) that under a contractionary monetary policy, the unique ergodic set of money holdings is such that markets shut down and therefore there is no monetary equilibrium. The reasoning behind the result is simple: due to the individual uncertainty, the agents need to hold large amounts of money to satisfy their tax needs. If an agent is relatively poor, she will fail to comply her tax obligations with positive probability, as her relative wealth will decrease as long as she is unproductive. We show that there is no stationary monetary equilibrium where tax obligations are always met. Under a contractionary policy, the wealth distribution is degenerate at a single value and consumption allocations are those of autarky. In [Section 3.2](#) we discuss the second best nature of the monetary policy. That is, we show that there is no policy able to complete the markets so that agents enjoy the consumption level of the complete markets allocation. We also show that generalizing the monetary policy rule to depend on the measure of wealth, which encompasses the history of shocks and transactions, does not affect this result.

To prove that an expansionary policy is optimal, [Section 4](#) characterizes analytically the equilibrium quantities and prices as a function of the growth rate of money and the distribution of money holdings. [Section 4.1](#) derives the invariant distribution of money holdings. The optimality of an expansionary policy is discussed in [Section 5](#), introducing an ex-ante welfare measure. It is assumed throughout that Inada conditions hold and that money is the only asset. Noting that the consumption of unproductive agents is zero under autarky implies that the ex-ante expected utility of contractionary policy, as well as that of an hyper-expansionary policy, approaches minus infinity as the unconditional probability of being unproductive is

strictly positive.³ To show that positive money growth rates dominates the constant money case we resort to numerical analysis. For strictly positive discount rates, the model is characterized by a single parameter which governs the length of unproductive spells. We vary this parameter and show that in every case the optimal money growth rate is strictly positive.

In [Section 6](#) we depart from the assumption of constant money growth rate to allow the government to do some state dependent policy. To do this we assume that the government observes the amount of money in the hands of unproductive agents but does not observe transactions nor identities. We find that the optimal policy expands the monetary base when unproductive agents are poor and contracts the monetary base otherwise. Interestingly, there is a bound to the size of the expansions as given by a strong connection of policy in the corners.

[Section 7](#) summarizes the main findings, discusses key differences with respect to related papers and future work.

2 The model

This section describes the model economy: agents' preferences, production possibilities, and markets. Two useful benchmarks are presented: the (efficient) allocation with complete markets and the optimal monetary policy with no uncertainty.

We consider two infinitely lived agents, indexed by $i = 1, 2$, and assume that at each point in time only one agent can produce. The productive agent transforms labor into consumption one for one, the unproductive agents cannot produce. The productivity of labor is state dependent: the duration of productivity spells is random, exponentially distributed, with mean duration $1/\lambda > 0$. Money, an intrinsically useless piece of paper, is distributed at each time t between the two agents so that $m_t^1 + m_t^2 = m_t$. The constant growth rate of the money supply is μ , so that the money supply follows $m_t = m_0 e^{\mu t}$ with m_0 given.

Let $\rho > 0$ denote the time discount rate, ω denote a history of shocks and money supply

³It is easily seen that the equilibrium allocation converges to autarky as inflation diverges to infinity.

levels, and $s(t, \omega) = \{1, 2\}$ be an indicator function denoting which agent is productive for a given history ω and current time t . Agent i chooses consumption c^i , labor supply l^i , and depletion of money balances \dot{m}^i , in order to maximize her (time-separable) expected discounted utility,

$$\max_{\{c^i(t, \omega), l^i(t, \omega), \dot{m}^i(t, \omega)\}_{t=0}^{\infty}} \mathbb{E}_0 \left\{ \int_0^{\infty} e^{-\rho t} [u(c^i(t, \omega)) - l^i(t, \omega)] dt \right\} \quad (1)$$

subject to the constraints

$$\dot{m}^i(t, \omega) \leq [l^i(t, \omega) + \tau(t, \omega) - c^i(t, \omega)] / \tilde{q}(t, \omega) \quad \text{if } s(t, \omega) = i \quad (2)$$

$$\dot{m}^i(t, \omega) \leq [\tau(t, \omega) - c^i(t, \omega)] / \tilde{q}(t, \omega) \text{ and } l^i(t, \omega) = 0 \quad \text{if } s(t, \omega) \neq i \quad (3)$$

$$m^i(t, \omega) \geq 0 \quad , \quad l^i(t, \omega) \geq 0 \quad , \quad c^i(t, \omega) \geq 0 \quad (4)$$

where $\tilde{q}(t, \omega)$ denotes the price of money, i.e. the inverse of the consumption price level, $\tau(t, \omega)$ denotes a government lump-sum transfer to each agent, and expectations are taken with respect to the processes s and m conditional on time $t = 0$.

A monetary policy with $\mu > 0$ is called expansionary, a policy with $\mu < 0$ is called contractionary. It is immediate that when the money supply is constant ($\mu = 0$) the economy is the one analyzed by [Scheinkman and Weiss \(1986\)](#). For any history ω the monetary policy μ determines the transfers to the agents τ_t through the government budget constraint,

$$\tilde{q}_t \mu m_t = 2\tau_t \quad (5)$$

which states that transfers are financed by printing money. The government transfer scheme implies that in the case of a contractionary policy agents must use their money holdings to pay taxes (i.e. $\tau < 0$). The next ‘‘tax solvency’’ constraint imposes this restriction

$$m_t^i \geq -\frac{\mu}{2} m_t \quad \forall t, \quad i = 1, 2 \quad . \quad (6)$$

Notice that the government cannot differentiate transfers across agent-types. This follows from the assumption that the identity of the productive type is not known to the government. [Levine \(1991\)](#) shows in a similar setup that, because of the hidden information assumption, the best mechanism is linear and can be understood as monetary policy. We extend on this in [Appendix C](#) where we explore the type of allocations that can be achieved using tax policy by changing government powers (commitment vs. no commitment), types of available taxes (lump-sum vs. distortionary), and government knowledge about the state (agent's type observable vs. not observable).

Next we state two important remarks. The first one characterizes a symmetric efficient allocation with complete markets (the proof is standard so we omit it):

Remark 1 *Assuming complete markets and an ex-ante equal probability of each state ($s = 1, 2$), the symmetric efficient allocation prescribes the same constant level of consumption, \bar{c} , for both agents, where \bar{c} solves $u'(\bar{c}) = 1$.*

Thus without borrowing constraints the efficient allocation solves a static problem, and it encodes full insurance: agents consume a constant amount (equal since we assume ex-ante identity) and the aggregate output is constant.

The next remark characterizes the optimal monetary policy in the case of no uncertainty. This helps highlighting the essential role of uncertainty in our problem. In particular, consider the case where each agent is productive for T periods, and then becomes unproductive for the next T periods. Without loss of generality, for the characterization of the stationary equilibria, let us assume that the economy starts in period $t = 1$ with agent 1 being productive and agent 2 owning all the money, so that $m_{t=1}^1 = 0$. We have

Remark 2 *Let $u(c^i) = c^{i1-\theta}/(1-\theta)$. The Euler equation for \dot{m}_t^i gives $\rho = \frac{\dot{q}_t}{q_t} - \theta \frac{\dot{c}_t^i}{c_t^i} - \mu$ which is solved by the “Friedman rule”: $\mu = -\rho$, and $\dot{c}_t^i = \bar{c}$ for $i = 1, 2$ and all t .*

See [Appendix B](#) for the proof. This remark, together with the efficient allocation described in [Remark 1](#), shows that without uncertainty this economy replicates [Townsend \(1980\)](#), [Bewley](#)

(1980) result on the optimality of the “Friedman rule”.

3 Monetary policy in the stochastic model

This section defines the monetary equilibrium in the original model with stochastic production opportunities. We look for a markovian equilibrium where $\tilde{q}(t, \omega) = \tilde{q}(m(t, \omega), m^i(t, \omega), s(t, \omega))$. With a slight abuse of notation this implies $c^i(t, \omega) = c^i(m(t, \omega), m^i(t, \omega), s(t, \omega))$, and $l^i(t, \omega) = l^i(m(t, \omega), m^i(t, \omega), s(t, \omega))$, and $\dot{m}^i(t, \omega) = \dot{m}^i(m(t, \omega), m^i(t, \omega), s(t, \omega))$. In other words, we are looking for an equilibrium that depends solely on three states: the level of the money supply, the distribution of money holdings, and the current state.

By a simple quantity theory argument, it is easily seen that the real variables of this economy everything are homogenous of degree one in the *level* of money, which acts as a numeraire. Hence we simplify the state space using that the nominal price of money is homogenous of degree minus one in the level of the money supply, i.e. $\tilde{q}(m, m^i, s) = \frac{1}{m}q(x^i, s)$, where $x \equiv \frac{m^i}{m}$ with $i = 1, 2$ and $s = 1, 2$. The variable $x \in [0, 1]$ is the share of total money balances in the hands of agent i , i.e. a measure of the wealth distribution. Likewise the consumption rule is homogeneous of degree zero in the level of the money supply $c^i(m, m^i, s) = c^i(x^i, s)$.

Without loss of generality, because of the symmetry of the environment, we will look at the equilibrium objects from the perspective of agent 1. We focus on equilibria where the relative price of money $q(x, 1) = q(1 - x, 2)$ depends on the share of money in the hands of the productive agent (but not on the *identity* of this agent). Next we define a monetary equilibrium.

Definition 1 *A monetary equilibrium is a price function $\tilde{q}(m, x, s) = \frac{1}{m} q(x, s)$, with $q : [0, 1] \times \{1, 2\} \rightarrow \mathbb{R}^+$ with $q(x, 1) = q(1 - x, 2)$, and a stochastic process $x(t, \omega)$ with values in $[0, 1]$, such that a consumer i maximizes expected discounted utility (equation (1)) subject to the budget constraints (2) and (3) with $q(t, \omega) = q(x, s)$, non-negativity (4), the government*

budget constraint (5) and the tax solvency constraint (6).

Solving the model requires characterizing the marginal value of money, given by the lagrange multiplier for m^i in the problem defined in (1): $\tilde{\gamma}(m, x, s)$. In particular, let $\tilde{\gamma}(m, x, 1)$ and $\tilde{\gamma}(m, x, 2)$ be the (un-discounted) multipliers associated to the constraints in [equation \(2\)](#) and [\(3\)](#), respectively, so that e.g. $\tilde{\gamma}(m, x, 2)$ measures the marginal value of money when the money supply is m , agent 1 holds a share x of it and she is unproductive. The first order conditions with respect to $l(t, \omega)$ and $c(t, \omega)$ give

$$\tilde{\gamma}(m, x, 1) = \tilde{q}(m, x, 1) \quad \text{and} \quad \tilde{\gamma}(m, x, 2) = \tilde{q}(m, x, 2) u'(c^1(x, 2))$$

As usual, these conditions equate marginal costs and benefits of an additional unit of money. For a productive agent the marginal benefit, $\tilde{\gamma}(m, x, 1)$, equals the cost of obtaining that unit, i.e. the disutility of work to produce and sell a consumption amount $\tilde{q}(m, x, 1)$, in nominal terms. For an unproductive agent the marginal cost is the value of the forgone unit of money $\tilde{\gamma}(m, x, 2)$, while the benefit is the additional units of consumption that can be bought with it, given by the product of the price q (consumption per unit of money) times the marginal utility of consumption. The homogeneity of $\tilde{q}(m, x, 1)$ with respect to m implies that the lagrange multiplier $\tilde{\gamma}$ is also homogenous, i.e. $\tilde{\gamma}(m, x, 1) = \gamma(x, 1)/m$. We can then rewrite the first order conditions in real terms as

$$\gamma(x, 1) = q(x, 1) \quad \text{and} \quad \gamma(x, 2) = q(x, 2) u'(c^1(x, 2)) \quad (7)$$

Notice that if $s(t, \omega) = 1$, then $c^1(t, \omega) = \bar{c}$, where \bar{c} solves $u'(\bar{c}) = 1$. It is shown in [Appendix D](#) that for $x \in (0, 1)$ the lagrange multipliers $\gamma(x, s)$ solve the following system of differential equations

$$\gamma_x(x, 1) \dot{x}(x, 1) = (\rho + \lambda + \mu)\gamma(x, 1) - \lambda\gamma(x, 2) \quad (8)$$

$$\gamma_x(x, 2) \dot{x}(x, 2) = (\rho + \lambda + \mu)\gamma(x, 2) - \lambda\gamma(x, 1) \quad (9)$$

The next lemma characterizes the functions $\gamma(x, 1)$ and $\gamma(x, 2)$ that solve this system:

Lemma 1 *Consider $x \in (0, 1)$, then $0 \leq \gamma(x, 1) \leq \gamma(x, 2)$, $\gamma_x(x, 2) < 0$, $\dot{x}(x, 2) < 0$, and $\lim_{x \downarrow 0} \dot{x}(x, 2) = 0$. Moreover $\frac{\partial c^1(x, 2)}{\partial x} > 0$.*

See [Appendix E](#) for the proof. The lemma characterizes the functions $\gamma(x, 1), \gamma(x, 2)$ in a monetary equilibrium. Intuitively, given the wealth share x , money is more valuable to unproductive than to productive agents. Moreover, since $\gamma(x, 2)$ is decreasing, the value of money for an unproductive agent is decreasing in wealth. Notice also that the consumption of an unproductive agent is increasing in his wealth x ; as the agent gets poorer he tries to avoid running out of resources by reducing her consumption. Finally, the lemma shows that in the case of an expansionary policy, for an unproductive agent it is indeed optimal to spend all the money transfer as $x \downarrow 0$, i.e. $\lim_{x \downarrow 0} \dot{x}(x, 2) = 0$.

3.1 No monetary equilibrium with contractionary policy

This section shows that with uncertainty there is no monetary equilibrium for any contractionary monetary policy. This result relates to [Bewley \(1983\)](#) who showed that in a neoclassical growth model with incomplete markets there is no monetary equilibrium if the interest rate is lower than the discount rate.⁴

A contractionary policy requires agents to pay lump sum taxes ($\tau < 0$). Hence agents commit to pay taxes which implies that $x_t \geq -\frac{\mu}{2}$ must be satisfied for all histories, as stated by [equation \(6\)](#). Consider the case where agent 1 has fraction of money balances x_t and the current state of the economy is $s(t, \omega) = 2$, which means that she is unproductive. If x_t is low enough, given that $\lambda > 0$ and finite, the agent will fail to comply with the monetary authorities with non-zero probability. On the other hand, consider the case where $x_t = 1$. In this case the agent is able to comply with her tax obligations with certainty. This implies that there exists a threshold \bar{x} such that for $x \geq \bar{x}$ the agent is able to cover her lifetime tax

⁴In his model agents face idiosyncratic shocks and there is a lump sum tax obligation that has to be covered by the agents.

needs with probability one. Note that the threshold must be independent of the current state s_t as with positive probability the states are reversed. In the next lemma we characterize this threshold.

Lemma 2 *If $\mu < 0$, for any state of the world $s(t, \omega)$, there is a unique threshold: $\bar{x} = 1/2$. Moreover, there is a unique ergodic set where $x_t = \frac{1}{2}$, $\forall t$.*

See [Appendix F](#) for the proof. Intuitively, given the uncertain duration of the productivity spell, the only value of money holdings that ensures compliance with tax obligations is $x = 1/2$. At this point, for any history of shocks, the identical lump-sum (negative) transfers reduce the money holdings of both agents proportionally, leaving the wealth distribution unaffected. This leads us to

Proposition 1 *Let $\mu < 0$: In the ergodic set there is no stationary monetary equilibrium and consumption allocations are autarkic.*

The proof of [Proposition 1](#) follows from noting that [Lemma 2](#) implies no trade in the ergodic set. Productive agents have an unsatisfied demand for money and unproductive ones have an unsatisfied demand for consumption goods.

3.2 The second best nature of monetary policy

In this section we show that there is no monetary policy μ where both types of agents consume the allocation of the complete markets economy. We state this in the next proposition.

Proposition 2 *If $\rho > 0$ and λ is finite, there is no value of μ such that $c^1(x, s) = c^2(x, s) = \bar{c} \forall s, x$.*

See [Appendix G](#) for a proof. In order to prove the proposition we start by noting that the complete markets allocation sustains constant price and marginal value of money. We then use this result to show that no monetary equilibrium in the economy with uncertainty can be

supported. When no monetary equilibrium exists, agents do not trade and therefore are not able to insure at all. This implies that they cannot attain constant consumption prescribed in the complete markets allocation which completes the proof.

To prove that no monetary equilibrium can be supported with a constant marginal value of money we note that this requires either a zero marginal value of money, and a zero price of money, or a contractionary policy at the discount rate $-\rho$. The first trivially cannot constitute a monetary equilibrium as one of the conditions for a monetary equilibrium is positive price level and marginal value of money (see [Definition 1](#) and [Lemma 1](#)). When $\mu = -\rho$ we know from [Section 3.1](#) that this cannot constitute a monetary equilibrium as we restricted the discount rate ρ to be positive.⁵ This last result contrasts with the optimal policy under known fluctuations (see [Remark 2](#)) where the complete markets allocation can be attained by deflating at the Friedman rate. The difference, as discussed in [Section 3.1](#), is that the length of unproductive spells is random here and therefore agents cannot trade in order to comply with their tax obligations. Both cases imply a collapse of the monetary economy resulting in no trade, money is valueless, and as a result there is no monetary equilibrium.

The state reversal parameter λ being finite is also important for the result to hold. Note that if $\lambda \uparrow \infty$, [equation \(8\)](#) provides that $\gamma(x, 1) = \gamma(x, 2)$, or that the marginal value of money is independent on whether the agent is productive or not. Furthermore, we can use this condition together with the first order conditions (see [equation \(7\)](#)) to obtain $c^1(x, 2) = 1$, and therefore the complete markets allocation is attained. Intuitively, this follows because insurance motives decrease to zero as agents are certain that their states are going to be reversed. **DISCUSS.**

Interestingly, this result does not hinge on μ being constant, as we state in the next corollary.

Corollary 1 *Let $\mu = \mu(x)$. If $\rho > 0$ and λ is finite there is no function $\mu(x)$ such that*

⁵If $\rho = 0$ the Friedman rule is attainable.

$$c^1(x, s) = c^2(x, s) = \bar{c} \quad \forall s, x.$$

The proof of the corollary is discussed in [Appendix G](#).

4 Expansionary monetary policy

In this section we study the case where $\mu > 0$. As in [Scheinkman and Weiss \(1986\)](#) we specialize to the case with logarithmic utility function, $u(c) = \ln(c)$, that will let us characterize analytically many results.⁶ We begin by completing the model solution by imposing the boundary conditions to the system of differential equations for the marginal value of money $\gamma(x, s)$, and describing a sufficient condition for a monetary equilibrium to exist. We also characterize some properties of the optimal consumption rule and we describe its local behavior when money growth and current money holdings x are sufficiently small: these results are useful for the main claim of the paper developed in [Section 5](#). With the same purpose, we characterize the invariant distribution of money holdings and explore some of its qualitative properties. Among them, we note that the density of money holdings has an asymptote as x approaches zero and we produce an approximation for the local behavior of the density for x “small enough”.

We start by discussing the boundary conditions for this problem. The boundaries concern the state in which the unproductive agent has no money. In this case unproductive agent spends the whole money transfer to finance her consumption. [Appendix E](#) gives a formal proof of this statement. The budget constraint gives $c_t^1 = \tau_t = q(0, 2)\mu/2$. Using [equation \(7\)](#) for the case of log utility gives

$$\gamma(0, 2) = \frac{2}{\mu} \quad \text{with} \quad \lim_{\mu \downarrow 0} \gamma(0, 2) = \infty \tag{10}$$

where the limit obtains because of Inada conditions. This is an important result in our

⁶Scheinkman and Weiss prove equilibrium existence and uniqueness for the log utility case. [Hayek \(1996\)](#) presents equilibrium existence and uniqueness results for the more general case of CRRA utility preferences with risk aversion greater than the log case.

analysis. An expansionary policy provides an upper bound to the marginal utility of money holdings because the price of money is always finite in an equilibrium and therefore the agent enjoys positive consumption even when she is very poor. When there is no money growth (i.e. as $\mu \downarrow 0$), the agent is not able to consume in poverty and therefore Inada conditions imply that the marginal utility of money approaches infinity.

The second boundary condition is also associated to the state in which the unproductive agent has no money and the productive agent has it all. Evaluating [equation \(8\)](#) at $x = 1$ with $\dot{x}(x, 1) = 0$ gives

$$\lambda \gamma(1, 2) = (\rho + \lambda + \mu) \gamma(1, 1) \quad (11)$$

The reason that x remains constant in this state is the following. The unproductive agent, who has a zero share of money, will spend all the money she receives from the government transfer in consumption goods. Intuitively, as the unproductive agent is impatient and there is a positive probability of becoming productive, saving a part of the transfer when $x = 0$ would be optimal if she expected to remain unproductive and that the value of money will go up in the future. In a rational expectations equilibrium, however, conditional on remaining unproductive this agent expects the value of money to go down. The amount of consumption goods bought by a unit of money, measured by $q(1 - x, 1)$, is increasing in x , the money holdings of the unproductive agent. Hence as $x \downarrow 0$, i.e. as the unproductive agent runs out of money, the real value of money falls.

We want to find expressions for the evolution of the share of outstanding money balances x over time. Consider the law of motion for the share of money held by type 1 when unproductive:

$$\dot{x}(x, 2) = \frac{\dot{m}^1}{m} - \frac{m^1}{(m)^2} \dot{m} = \mu \left(\frac{1}{2} - x \right) - \frac{c^1(x, 2)}{q(x, 2)} = \mu \left(\frac{1}{2} - x \right) - \frac{1}{\gamma(x, 2)} \quad (12)$$

where we used the budget constraint of the unproductive agent, [equation \(3\)](#), the government budget constraint, [equation \(5\)](#), and the first order condition in [equation \(7\)](#). Noting that

$\dot{x}(x, 1) + \dot{x}(1 - x, 2) = 0$ provides

$$\dot{x}(x, 1) = \mu \left(\frac{1}{2} - x \right) + \frac{1}{\gamma(1 - x, 2)} . \quad (13)$$

These equations show that the optimal change of real money holdings, on top of the government transfer, depends on the value of money for the unproductive agent relative to the value of money for the productive agent: $\gamma(x, 2)/q(x, 2) = \gamma(x, 2)/q(1 - x, 1)$. Notice from the first order conditions that the smaller is the consumption level of the unproductive agent, the higher is the value of $\gamma(x, 2)/q(x, 2)$, hence the smaller will be the (absolute) real value of money transfers.

For expositional purposes we rewrite the system of ODEs by substituting in for $\dot{x}(x, s)$ the corresponding expressions from [equation \(8\)](#) and [\(9\)](#),

$$\gamma_x(x, 1) \left[\mu \left(\frac{1}{2} - x \right) + \frac{1}{\gamma(1 - x, 2)} \right] = (\rho + \lambda + \mu)\gamma(x, 1) - \lambda\gamma(x, 2) \quad (14)$$

$$\gamma_x(x, 2) \left[\mu \left(\frac{1}{2} - x \right) - \frac{1}{\gamma(x, 2)} \right] = (\rho + \lambda + \mu)\gamma(x, 2) - \lambda\gamma(x, 1) \quad (15)$$

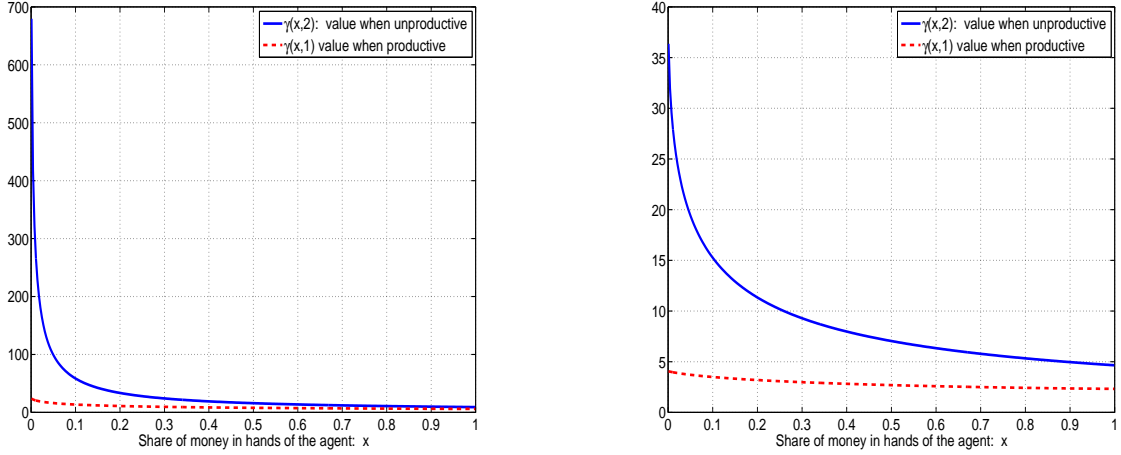
where the system's boundary conditions were given in [equations \(10\)](#) and [\(11\)](#). This is a system of delay differential equations.⁷

To illustrate the solution we present a few plots of the important objects in this economy. Throughout the paper we assume that the baseline case uses $\lambda = 0.1$ and $\rho = 0.05$. The parameter choices, and the graphical analysis, is solely for expositional purposes. In [Figure 1](#) we present the numerical solution to the system of delay ODEs that provide the marginal value of money. The left panel plots these values assuming $\mu = 0.002$ and the right panel assumes $\mu = 0.05$. The properties listed in [Lemma 1](#) are clearly seen. Also, the value of money when unproductive with zero wealth (i.e. $x = 0$) is much higher for the case where $\mu = 0.002$, as prescribed by [equation \(10\)](#). It can be seen that the properties described in

⁷Notice in particular that the delay is non-constant, which prevents an analytical solution in closed form. Notice however that the functions $\gamma(x, 1), \gamma(x, 2)$ are analytical, and that the above system allows to completely characterize these functions given initial values for $\gamma(\frac{1}{2}, 1), \gamma(\frac{1}{2}, 2)$.

Lemma 1 are satisfied.

Figure 1: The value of money as function of the wealth distribution x : $\gamma(x, 1), \gamma(x, 2)$



Parameters: $\lambda = 0.10$, $\rho = 0.05$. Left panel: $\mu = 0.002$. Right panel: $\mu = 0.05$

We now discuss the optimal consumption when unproductive. This is an important object because unproductive agents need to spend money to consume and therefore monetary policy will have a strong impact on their consumption behavior. Notice that

$$c^1(x, 2) = \frac{\gamma(1-x, 1)}{\gamma(x, 2)} \quad \text{with} \quad \frac{\partial c^1(x, 2)}{\partial x} > 0 \quad (16)$$

which follows from Lemma 1. The fact that consumption when unproductive is increasing in money holdings, implies that for any given money growth rate $\mu > 0$ consumption is smallest when the agent money holdings are zero, in particular at $x = 0$ we have

$$c^1(0, 2) = \frac{\mu}{2} \gamma(1, 1) \quad , \quad \text{with} \quad \lim_{\mu \downarrow 0} \frac{\mu}{2} \gamma(1, 1) = 0 \quad (17)$$

where the limit obtains from the agent's budget constraint. This is important for the welfare analysis that will follow because it shows that monetary transfers provide the unproductive agent with a lower bound to the consumption level. Without transfers, an agent with no money cannot consume.

A plot of the consumption function is given in [Figure 2](#). For $\mu = 0.002$, it can be seen that consumption when unproductive goes to zero as $x \downarrow 0$. Note also that the consumption when unproductive may be above the consumption when productive (i.e. $c = 1$) for a high enough wealth. The economics of this result is that as $x \uparrow 1$ the productive agent owns almost zero money, and is eager to accumulate money to insure against the possibility of a switch of the state. So the price of money is high (i.e. the price of consumption is low), and the productive agent is willing to produce a lot to refill his low money holdings.⁸ A comparison of the consumption function at a low and a high money growth (the two curves in the picture) illustrates the tradeoff of inflation: increasing the money growth rate (from 0.2 to 5 percent in the figure) provides higher consumption to relatively poor unproductive agents but decreases the consumption of unproductive agents with “high” wealth. An optimal choice of the money growth rate trades off these effects.

We are also interested in understanding how consumption behaves when the unproductive agent is very poor. The next lemma addresses this issue.

Lemma 3 $\lim_{x \downarrow 0} \frac{c_x^1(x,2)x}{c^1(x,2)} = Q_1$, where $Q_1 = 1 + \frac{\rho}{\mu} + \frac{\lambda}{\mu} \left(1 - \frac{\gamma(0,1)}{\gamma(0,2)}\right) > 1$ and finite $\forall \mu > 0$.

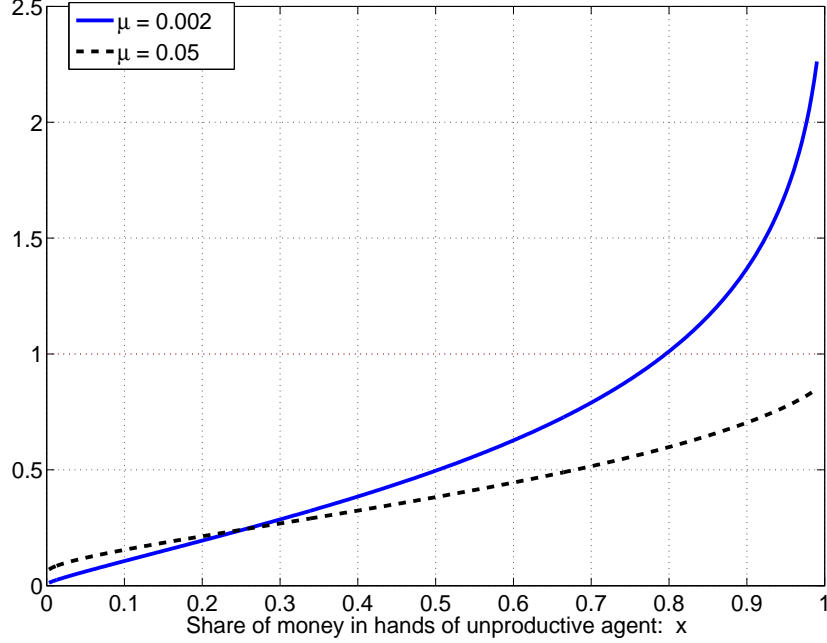
See [Appendix H](#) for the proof. [Lemma 3](#) states that when money holdings x is “small”, the unproductive agent’s elasticity of consumption with respect to money holdings approaches a value that is larger than one. This implies that for an unproductive agent that is sufficiently poor, her level of consumption decreases at a higher rate than her share of money holdings x . **ADD INTUITION.**

4.1 The invariant distribution of money holdings

Let $F(x, s)$ denote the CDF for the share of money holdings in state s with density $f(x, s) = \frac{\partial F(x, s)}{\partial x}$. The density function of the invariant distribution is derived from the usual Kol-

⁸If we let x be the wealth in the hands of the unproductive agent, the relative price of money in terms of consumption units is given by $\gamma(1 - x, 1)$, which is increasing in x (see [Figure 1](#) for a plot of the $\gamma(x, 1)$ function). It can be obtained that $c^1(1, 2) = \frac{\gamma(0,1)}{\gamma(1,1)} \frac{\lambda}{\rho + \lambda + \mu}$. As $\frac{\gamma(0,1)}{\gamma(1,1)} > 1$, this equation implies that there are values of $\frac{\lambda}{\rho + \lambda + \mu}$ such that $c^1(1, 2) > 1$.

Figure 2: Consumption when unproductive as function of the wealth distribution x



Parameters: $\lambda = 0.10$, $\rho = 0.05$.

mogorov Forward Equation (KFE) after imposing for stationarity. [Appendix I](#) derives the KFE for our model with poisson jumps in the state, which gives

$$0 = f_x(x, s_i)\dot{x}(x, s_i) + f(x, s_i)\frac{\partial \dot{x}(x, s_i)}{\partial x} + \lambda [f(x, s_i) - f(x, s_{-i})] \quad (18)$$

where $s_i = 1, 2$ denotes the current state and s_{-i} the other state. It is immediate that the densities satisfy $f(x, 2) = f(1 - x, 1)$ since, given the assumed symmetry of the shocks, for each agent with money $x = x'$ and $s = 2$ there is another agent with money $1 - x'$ and $s = 1$. This property allows us to concentrate the analysis on only one density: $f(x, 2)$. Using the expressions for $\dot{x}(x, s)$ derived in [equation \(12\)](#) and [equation \(13\)](#) gives

$$\frac{f_x(x, 2)}{f(x, 2)} = \frac{\gamma(x, 2) (\lambda - \mu) - \frac{(\rho + \lambda + \mu)\gamma(x, 2) - \lambda\gamma(x, 1)}{1 + \mu(x - \frac{1}{2})\gamma(x, 2)}}{1 + \mu(x - \frac{1}{2})\gamma(x, 2)} - \frac{\lambda}{\mu(\frac{1}{2} - x) + \frac{1}{\gamma(1-x, 2)}} \equiv \Omega(x) \quad (19)$$

It follows that

$$f(x, 2) = C e^{\int_{1/2}^x \Omega(z) dz} \quad \text{where} \quad C = \left[\int_0^1 e^{\int_{1/2}^x \Omega(z) dz} dx \right]^{-1} \quad (20)$$

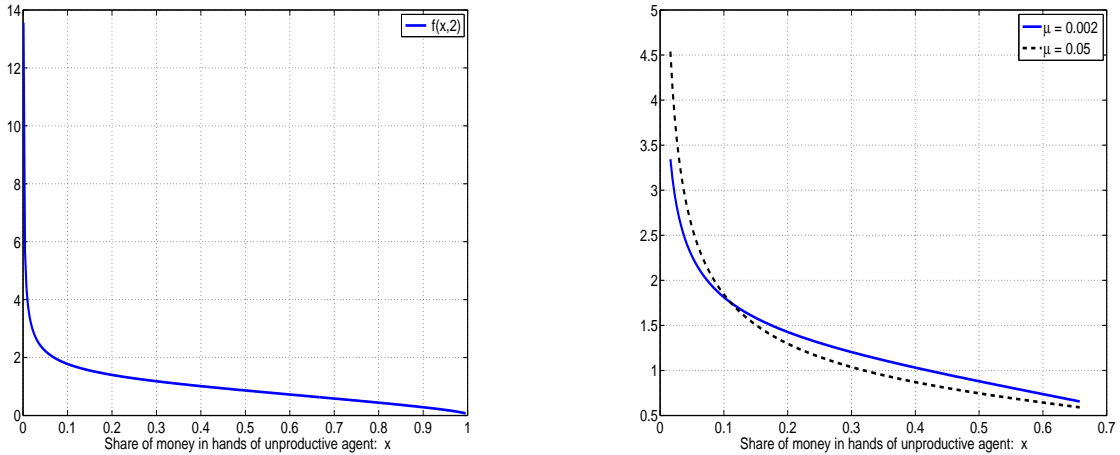
where the constant C ensures that $\int_0^1 f(x, 2) dx = 1$.

The next lemma establishes useful properties of the density function for money holdings:

Lemma 4 *Properties of the invariant density function $f(x, 2)$ for $x \in (0, 1)$: (i) continuous and differentiable in x , (ii) $\lim_{x \downarrow 0} f(x, 2) = +\infty$, and (iii) $\lim_{x \downarrow 0} f_x(x, 2) = -\infty$.*

The details of the proof can be found in [Appendix J](#). [Figure 3](#) plots the density function $f(x, 2)$ for two parametrization. In the left panel it can be seen that, as shown in [Lemma 4](#) the density function has an asymptote as the share of wealth in hands of an unproductive agent approaches zero. In the right panel we show what happens with the mass “near” zero as we change the money growth rate. As expected, increasing the money growth rate μ increases the amount of histories where unproductive agents have little money. This follows because a higher μ implies higher insurance, and therefore a lower cost of running out of money.

Figure 3: Invariant distribution of wealth $f(x, 2)$



Parameters: $\lambda = 0.10$, $\rho = 0.05$, (left panel: $\mu = 0.002$).

Remark 3 *The density $f(x, 2)$ is continuous in μ .*

Remark 3 follows as from equation (19) we can see that $\frac{f_x(x,2)}{f(x,2)}$ is a continuous function of growth rate of money μ because it is a composition of continuous functions on μ . This remark, together with Lemma 4, implies that, for any μ the density of unproductive agents has an asymptote at infinity as x converges to zero from above.

5 On the optimality of monetary policy

In this section we show that a finite expansionary policy is optimal. We do this by a combination of analytical and numerical analysis. We show analytically that both contractions and extreme expansions (i.e. $\mu \uparrow \infty$) are not optimal from an ex-ante perspective. Then, using numerical methods, we show that a strictly positive and finite inflation rate maximizes ex-ante welfare.

We view the problem of the optimal policy from an ex-ante perspective and we let the monetary authority choose the growth rate of money μ to maximize the ex-ante expected utility measure

$$\begin{aligned} \mathcal{W}(\mu) &= \mathbb{E}_x \{ u(c^1(x, 2; \mu)) + u(c^2(1-x, 2; \mu)) - l^2(1-x, 2; \mu) \mid \mu \} \\ &= \int_0^1 f(x, 2; \mu) [u(c^1(x, 2; \mu)) + u(c^2(1-x, 2; \mu)) - l^2(1-x, 2; \mu)] dx \end{aligned}$$

where the notation emphasizes that the consumption paths and the probability density of money holdings depend on the money growth rate μ . The expression for $\mathcal{W}(\mu)$ measures the stationary ex-ante (expected) utility, i.e. the welfare of any given agent before her initial state is realized. Types are given equal weights because the symmetry of the Markov process for the shocks implies that agents are productive 1/2 of the time. It is assumed that initial money holdings x are drawn from the invariant distribution $f(x, 2)$.⁹

⁹Other arguments in favor of this criterion can be found in Aiyagari and McGrattan (1998).

The next lemma shows that ex-ante expected utility diverges as the money growth rate approaches infinity or when a contractionary monetary policy is followed.

Lemma 5 $\lim_{\mu \uparrow \infty} \mathcal{W}(\mu) = -\infty$. Also, for $\mu < 0$, $\mathcal{W}(\mu) \downarrow -\infty$.

The proof of the lemma is given in [Appendix K](#). The intuition behind the lemma is that when money grows really fast (i.e. $\mu \uparrow \infty$) the marginal value of a unit of money becomes zero because the monetary policy wipes out completely the value of accumulated money holdings: such a policy destroys all the incentives for a productive agent to accept money in exchange for goods. In fact, independently of the distribution of outstanding money holdings at the beginning of the period x , this policy is such that the distribution of money holdings across agents is constant at $x = 1/2$, with all the mass concentrated on this point. In this case, as it is also true for the contractionary case where $\mu < 0$, the economy is in autarky and no trade occurs. Therefore, because ex-ante agents are unproductive 1/2 of the time and in this state their consumption is zero, the ex ante utility diverges to $-\infty$ in both of these cases.

For finite non-negative values of money growth rate μ the ex-ante expected welfare $\mathcal{W}(\mu)$ is finite.¹⁰ To show that the optimal policy is expansionary relative to the constant money case we resort to numerical analysis. Note that as long as the discount rate ρ is positive¹¹, it does not play any particular role in the choice of the optimal policy.¹² Therefore, numerically the model can be understood as a single parameter model, λ . Then, we will look at the optimal choice of μ as we vary λ .

We show that the optimal μ is positive in two different ways. First, we show that for a given value λ , the objective function $\mathcal{W}(\mu)$ is single peaked at positive values of μ . Second, we look at the elasticity of the objective function $\mathcal{W}(\mu)$ evaluated at $\mu = 0$ and we show that as we vary λ the local elasticity is always positive. This result, together with [Lemma 5](#)

¹⁰That $\mathcal{W}(0)$ is finite follows immediately from [Scheinkman and Weiss \(1986\)](#). For $\mu > 0$ note that $\mathcal{W}(\mu) > \ln\left(\frac{\mu}{2}\gamma(1, 1)\right) \int_1^0 f(y, 2)dy > -\infty \forall 0 < \mu < \infty$, where $\frac{\mu}{2}\gamma(1, 1) = c^1(0, 2)$.

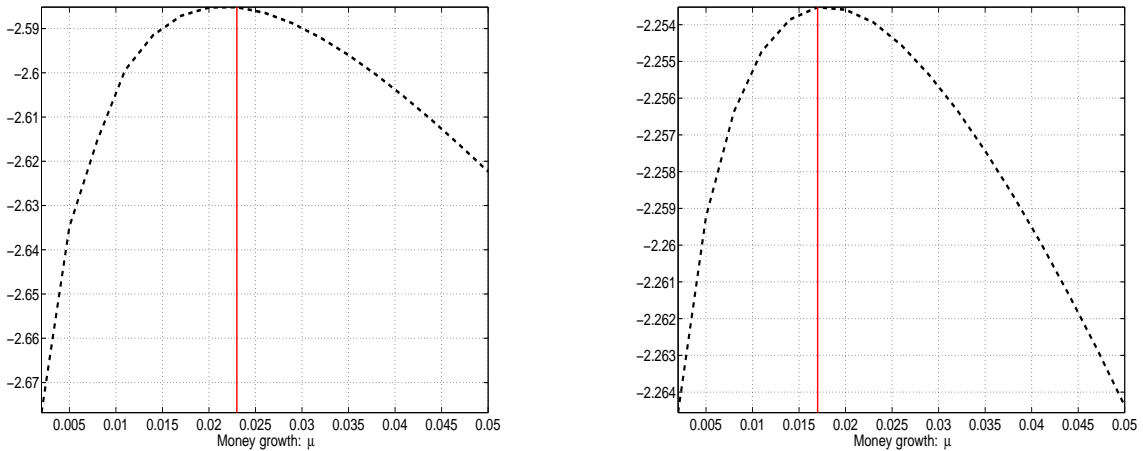
¹¹ $\rho = 0$ implies that the Friedman rule is attainable.

¹²It will, of course, affect the level of the policy but it does not affect the comparison between zero and positive money growth rates.

which can be read as stating that the derivative of the objective function $\mathcal{W}(\mu)$ is negative for $\mu \uparrow \infty$, shows that the optimal level of money growth rate is strictly positive and finite.

In [Figure 4](#) we show that, once we fix λ , ex-ante expected welfare $\mathcal{W}(\mu)$ is a concave function of the money growth parameter μ , attaining its peak at positive values of μ . Moreover, it can be seen that increasing λ makes the optimal value of the money growth rate μ to decrease. Intuitively, an increase in λ implies a decrease in the expected length of unproductive spells, $1/\lambda$, so that insurance motives become less relevant relative to production incentives, therefore providing a rationale for lowering the production distortion.¹³

Figure 4: Ex-ante expected welfare $\mathcal{W}(\mu)$



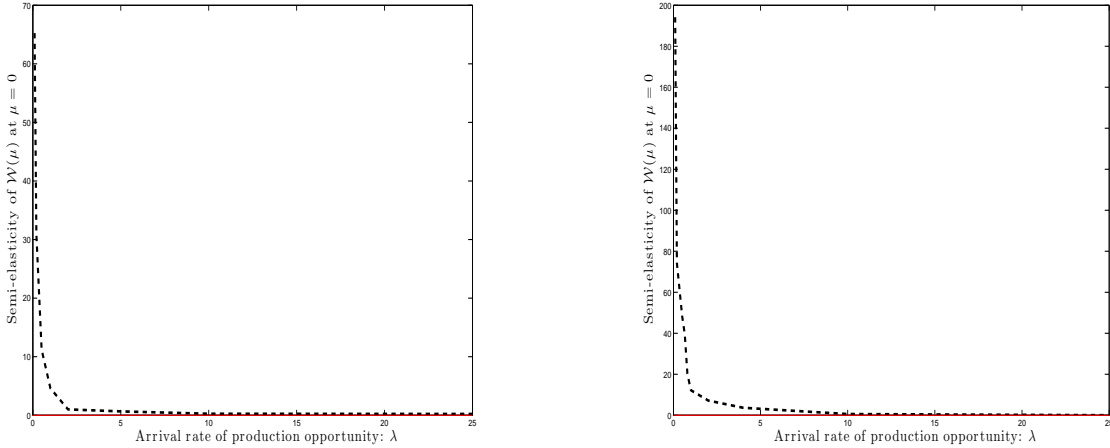
Parameters: $\rho = 0.05$. Left panel: $\lambda = 0.10$. Right panel: $\lambda = 0.50$.

We want to show that the the optimal money growth rate is positive for a wide range of values of λ . We do this by analyzing the first derivative of the objective function with respect to money growth rate parameter μ , $\frac{\partial \mathcal{W}(\mu)}{\partial \mu}$, which is possible because $\mathcal{W}(\mu)$ is continuous and differentiable in μ for all $0 < \mu < \infty$. [Lemma 5](#) provides that $\lim_{\mu \uparrow \infty} \frac{\partial \mathcal{W}(\mu)}{\partial \mu} < 0$. This result, together with differentiability, implies that if $\frac{\partial \mathcal{W}(0)}{\partial \mu} > 0$ the optimal inflation rate is strictly positive and finite. In other words, the ex-ante expected utility $\mathcal{W}(\mu)$ attains its maximum at some $0 < \mu < \infty$.

¹³In [Section 3.2](#) we show that the complete markets allocation can be attained as $\lambda \uparrow \infty$ because insurance motives vanishes completely.

In [Figure 5](#) we present the semi-elasticity of the ex-ante expected welfare $\mathcal{W}(\mu)$ evaluated at $\mu = 0$ as a function of λ . It can be seen in the plot that the semi-elasticity is positive for every value of λ so that the optimal money growth rate is strictly positive. Note that this result is independent on the value of ρ , as shown in the two different panels.

Figure 5: Semi-elasticity of ex-ante expected welfare $\mathcal{W}(\mu)$ at $\mu = 0$



Left panel: $\rho = 0.05$. Right panel: $\rho = 0.25$.

The economics behind the optimal money growth rate being positive are simple: in an economy with borrowing constraints agents will occasionally incur into histories, i.e. long spells of low-productivity, in which they "almost" run out of money. The positive money transfers the government distributes provide a floor to how bad consumption looks in these states. But this provision of insurance comes with a cost as productive agents are less willing to accept money balances in exchange for goods as monetary expansions decrease the return of money. The optimal finite money growth rate strikes a balance between these opposing forces.

6 State dependent policy - WORK IN PROGRESS

The analysis of the optimal policy assumed that money growth μ was constant. This section explores the consequences of allowing the government to use a state dependent policy. We let the government tie the monetary policy to the distribution of wealth. In particular, we allow the policy to be dependent on the share of current money held by unproductive agents,

$$\mu = \mu[x]$$

this policy is such that the government observes the share of wealth held by unproductive agents, but cannot identify individual agents nor observes transactions. This implies that the complete markets allocation, where both types of agents consume \bar{c} for every value of wealth x and state of nature s , cannot be attained (see [Section C](#) and [Section 3.2](#)).

When we allow the monetary policy to depend on the distribution of wealth we need to update the system of ODEs governing the marginal values of money $\gamma(x, s) \quad \forall x, s$, the boundary conditions, and the stationary density of money holdings $f(x, 2)$. The system of ODEs, presented in [equations \(14\)](#) and [\(15\)](#), become

$$\begin{aligned}\gamma_x[x, 1]\dot{x}[x, 1] &= (\rho + \lambda + \mu[1 - x])\gamma[x, 1] - \lambda\gamma[x, 2] \\ \gamma_x[x, 2]\dot{x}[x, 2] &= (\rho + \lambda + \mu[x])\gamma[x, 2] - \lambda\gamma[x, 1]\end{aligned}$$

where

$$\begin{aligned}\dot{x}(x, 1) &= \mu(1 - x) \left(\frac{1}{2} - x \right) + \frac{1}{\gamma(1 - x, 2)} \\ \dot{x}(x, 2) &= \mu(x) \left(\frac{1}{2} - x \right) - \frac{1}{\gamma(x, 2)}\end{aligned}$$

and boundary conditions

$$\begin{aligned}\gamma(0, 2) &= \frac{2}{\mu(0)} \\ \lambda\gamma(1, 2) &= (\rho + \lambda + \mu(0))\gamma(1, 1) .\end{aligned}$$

The tax-solvency constraint in [equation \(6\)](#) implies that the money rule must satisfy

$$\mu(x) \geq -2x \quad .$$

The inverse of the ratio of the density function $f(x, 2)$ to its first derivative also changes,

$$\begin{aligned}\frac{f_x(x, 2)}{f(x, 2)} &= \frac{\gamma(x, 2) \left(\lambda - \mu(x) + \left(\frac{1}{2} - x\right) \mu_x(x) \right) - \frac{(\rho + \lambda + \mu(x))\gamma(x, 2) - \lambda\gamma(x, 1)}{1 + \mu(x) \left(x - \frac{1}{2}\right) \gamma(x, 2)}}{1 + \mu(x) \left(x - \frac{1}{2}\right) \gamma(x, 2)} \\ &\quad - \frac{\lambda}{\mu(1 - x) \left(\frac{1}{2} - x\right) + \frac{1}{\gamma(1 - x, 2)}}\end{aligned}$$

The optimal policy trades off insurance motives and production incentives. Making the policy to depend on the distribution of wealth provides some degree of freedom to the government as it is able to decouple (at least in some degree) these two forces. To make this point clear we explore numerically a case where

$$\mu(x) = \mu_1 + (\mu_0 - \mu_1) (e^{-\kappa x} - x e^{-\kappa})$$

with $\mu(0) = \mu_0$, $\mu(1) = \mu_1$, and $\mu(x)$ non increasing and convex. Note also that if $\kappa = 0$ then $\mu(x)$ is linear, and if $\kappa = 0$ and $\mu_0 = \mu_1$ then $\mu(x)$ is constant in the whole range.

We compare the best constant μ policy with the best linear and best non-linear policies (NOTE OF CAUTION: this is work in progress... we used a very small grid to find the best policies... this will be improved soon). The best linear policy attains very similar ex-ante expected utility than the best constant policy showing that apparent extra degree of freedom is not able to decouple the incentives to produce from the insurance motives. The non-linear

case exhibits an important increase in ex-ante welfare (recall that ex-ante welfare in the complete markets case is -2) by printing lots of money when the unproductive agent is very poor but deflating the economy for almost every other level of wealth (see [Figure 7](#)).

Table 1: Expected inflation and money growth rates under different policy rules

Type of policy	$E[\pi]$	$E[\mu]$	μ_0	μ_1	κ	$E[\mathcal{W}]$
Constant	5.2%	2.3%	0.023	0.023	0	-2.75
Linear	15.3%	7.8%	0.1425	-0.05	0	-2.74
Non-linear	7.5%	-4.5%	0.1425	-0.05	100	-2.26

π : inflation rate (see [Appendix ??](#)). To compute expected values we used the density of money holdings $f(x, 2)$. The functional form for the money growth rate is $\mu(x) = \mu_1 + (\mu_0 - \mu_1)(e^{-\kappa x} - xe^{-\kappa})$.

Figure 6: Different policy rules

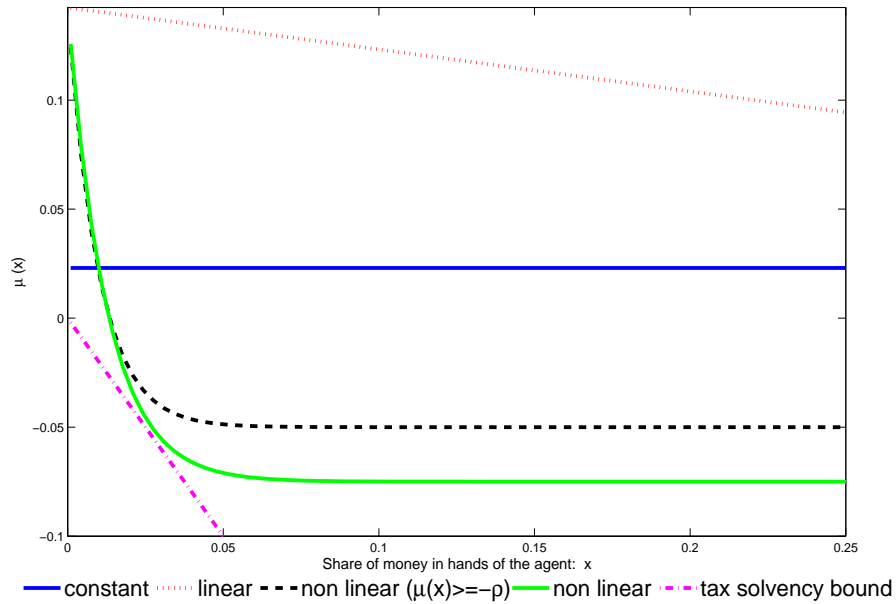
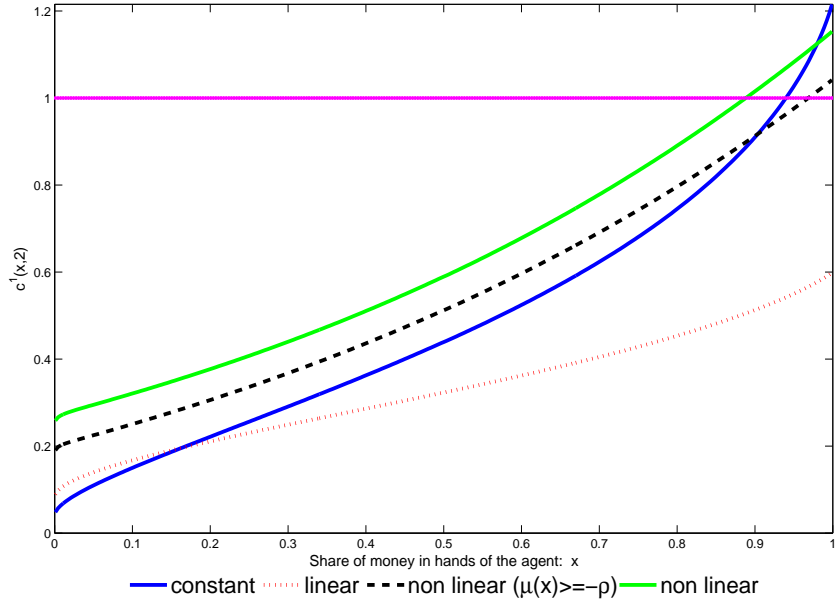


Figure 7: Different policy rules



7 Concluding remarks and future work

This paper analyzed the optimal anticipated money growth rule in the [Scheinkman and Weiss \(1986\)](#) economy. The choice involves a tradeoff between insuring agents who are hit by long sequences of bad shocks vs. minimizing the distortion that the inflation tax induces on production decisions. We showed that a monetary policy involving deflation, such as the Friedman rule, or an infinite inflation, do not support a monetary equilibrium, and leads to autarky. With a constant money supply the government provides no insurance: we showed that this corner solution is not optimal; from an ex-ante perspective, agents are better off with strictly positive money growth. The presence of uninsurable risk is key to this result: in the non-stochastic of our economy the Friedman rule is optimal.

Two important assumptions behind the optimality of the expansionary monetary policy are that the marginal utility of consumption approaches infinity as consumption approaches zero (Inada conditions) and that unproductive agents can only consume if their money bal-

ances are not zero (e.g. no home production is allowed). Under these assumptions, the expansion of the money supply is good because the precautionary savings motive that follow from the unboundedness of the utility function dominates the cost of expansions (decrease in production). In other words, providing some extra cash to poor agents is always optimal even though it implies a decrease in aggregate production. If we relax either assumption, e.g. assume that the utility function is bounded or extend the model to allow for positive endowments, an expansionary monetary policy might not be optimal anymore as the precautionary savings motive can be dominated by the inflation effect on production. [Kehoe, Levine and Woodford \(1990\)](#) explore a particular case of these economies, one where there are constant endowments, no production, and where agents differ in their marginal utility of consumption. Because agents can trade any fraction of their endowment, they can always overcome Inada conditions by simply keeping a fraction of their endowment for personal consumption. They show numerically that the optimality of an expansionary policy depends on parameter values and therefore is not guaranteed.

What would happen to optimality in our model if we modify either of the assumptions is not clear. But we conjecture that our results are robust to the introduction of small endowments. We argue that this is the case as our proof hinges on showing that an expansionary monetary policy dominates constant and contractionary monetary policies. Note that in our benchmark model endowments are zero. Because of continuity this implies that small endowments should not reverse the result, even though they provide some insurance

Another issue that we see as interesting for future research is to numerically investigate the properties of optimal state dependent rules. In this model we did not allow the money growth to vary with the business cycle. Intuitively, it would seem reasonable that a government wants to expand the money supply in a recession (e.g. when x is small), but may want to deflate when x is more evenly distributed. We leave this topic for future research.

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Appendix

A Related Literature

A few previous contributions discuss environments where a flat monetary expansion is efficient in an economy with incomplete markets and where money serves an essential role. The seminal paper in this line is [Levine \(1991\)](#), who considers an endowment economy where the agents' utility function change randomly according to whether they are “buyers” or “sellers”, a state that follows an exogenous Markov process. Levine's develops an analytical argument which shows that an expansionary monetary policy can attain the first best. Three assumptions are crucial for this result: uncertainty, constant individual endowment, and bounded utility functions. Uncertainty is key because, as in [Scheinkman and Weiss](#) and our model, it creates a demand for insurance on the part of “unlucky” agents. Constant endowment implies that monetary policy, through either inflation or deflation, cannot affect production decisions, which simplifies the analysis. In Levine's model sellers sell their entire endowment, which amounts to a restriction on the agents marginal utilities of the set of feasible trades. Because of this assumption monetary policy can provide insurance at no cost since altering the relative price has no effect on welfare in the corner solution.

[Kehoe, Levine and Woodford \(1990\)](#) extend Levine's setting to allow for internal solutions where sellers do not necessarily sell all their endowment. For reasons of tractability, they restrict attention to equilibria in which what happens in each period is *independent of history* (two state markov equilibria). Under these assumptions the distribution of money is degenerate: “sellers” always end the period owning *the whole* money stock. In this setup optimal monetary policy has a cost (it distorts the sellers' choices) and a benefit, namely that it provides some insurance. Optimal monetary policy can be either expansionary or contractionary. They use numerical solutions of the model to study how the optimal inflation varies with parameters' values.¹⁴

Compared to these papers, our contribution is to analyze the question of the optimal policy in the context of a production economy, and to focus, as [Scheinkman and Weiss](#), on equilibria in which the decisions in each period depend on the whole history of shocks, as summarized by the distribution of money holdings. We analytically characterize the non-degenerate distribution of money holdings, and provide a characterization of the the ex-ante efficient anticipated monetary policy. The concluding section of our paper discusses some key assumptions that explain the differences in results between these papers.

Our work is also related to [Imrohoroglu \(1992\)](#) who evaluates the welfare costs of inflation in an economy with borrowing constraints and where money is essential for trading. In the model a continuum of agents are faced with idiosyncratic shocks with no aggregate uncertainty that define if agents are employed, getting goods y , or unemployed, getting θy with $\theta < 1$. The consumption goods are homogeneous and therefore agents want to trade

¹⁴ [Green and Zhou \(2005\)](#) also study a Bewley-type setup using mechanism design theory. They restrict the feasible set of policies to model monetary equilibria and compare the allocations obtained under alternative “institutional mechanisms”. One of their examples features agents who differ in their marginal utilities and endowments which vary randomly, as in Levine's. It is shown that, because utility functions are bounded, the optimal allocation is to assign all the consumption to agents with high marginal utility, which can be done through an expansionary monetary policy.

to smooth consumption. In order to do so, they are endowed with some fraction of money and trade in a centralized market. For a calibration of the model, Imrohorglu estimates that an expansionary monetary policy is costly. The calibrated value for θ is crucial for this result (θ is calibrated to be 1/4 from previous work by her on the topic). For small enough values for θ the marginal utility of consumption when unemployed would become arbitrarily large and therefore inflations, that serve as a transfer to poor agents, would become optimal (in [Section L](#) we discuss a similar model and show that a strictly positive inflation rate is optimal). Our model departs from Imrohorglu's in several ways. First, she has an endowment economy while we deal with a production economy which adds an extra margin where inflation plays a role. Moreover, in her model when an agent is unemployed receives some endowment that overturns Inada conditions, while in our model an unproductive agent receives nothing so Inada conditions become of first order importance. It follows that in our setup a monetary expansion, that among other implies a transfer from rich to poor agents, provides insurance to unproductive agents. That is, an expansion can be understood as the provision of some endowment to the unlucky agents. Second, her assumption that shocks are idiosyncratic together with a centralized market for goods implies that the price of money is independent of the distribution of money holdings. In our model, because we are dealing with agents which shocks are correlated, the price of money is not orthogonal to the distribution of money holdings. Therefore, even though both economies face no aggregate uncertainty, her economy exhibits a constant steady state while ours has a stationary distribution with cycles.

A related question on an optimal scheme for monetary transfers in a search theoretic framework is evaluated in [Berentsen, Camera and Waller \(2005\)](#), who show that a one-time money injection may increase ex-ante welfare. Their analysis is done in the search model of money of [Lagos and Wright \(2005\)](#), where a non-degenerate distribution of money holdings within a period provides a role for monetary policy. An important difference with our setup is that here the distribution of wealth at the end of the period is degenerate at a single value and therefore the monetary policy has no effect on the dynamics of the economy. This difference is important if we want to allow the monetary authority to choose the money growth. As noted by Berensten, Camera, and Waller, in the long run money is neutral in their model and therefore the optimal money growth is to deflate following Friedman's prescription.¹⁵

In a broad sense the mechanism that we study is related to a recent strand of literature dealing with the interplay between the agents' liquidity and the business cycle, these include [Kiyotaki and Moore \(2008\)](#), [Guerrieri and Lorenzoni \(2009\)](#), [Chamley \(2010\)](#), [Brunnermeier and Sannikov \(2011\)](#). Our results are also related to [Aiyagari \(1995\)](#) who studies a neoclassical

¹⁵ Another related papers is [Bhattacharya, Haslag and Martin \(2005\)](#), which studies several environments where the monetary policy can have redistributive effects and show that the Friedman rule might not be *ex-post* optimal. In particular, related to our work, they show that in an example of [Townsend \(1980\)](#) deterministic Turnpike model ex-post optimality is not attained by the Friedman rule. The different welfare criterion (ex-post versus ex-ante) is key here: [Remark 2](#) in the next section shows that the Friedman rule is ex-ante optimal in a deterministic version of our economy. [Edmond \(2002\)](#) and [Akyol \(2004\)](#) also discuss the possibility of an efficient expansionary policy. Both papers study an endowment economy and use numerical simulations to show that positive inflation may be optimal. A main difference between these papers and ours (and Levine's) is that money in these papers serves no essential role, it is valued only because it enters the utility function of agents. This difference is important because, as explained by [Kehoe, Levine and Woodford \(1990\)](#), it does not allow the redistributive consequences of monetary policy to be analyzed.

growth model with borrowing constraints, as in [Bewley \(1977\)](#). Aiyagari shows that a positive capital tax rate is optimal because agents are over-saving due to precautionary motives and therefore the capital stock is too high. Because the government faces no incomplete markets, an optimal policy involves taxing capital and using the proceedings to produce a public good that all consumers enjoy, independently of their idiosyncratic shock. Our finding that the optimal policy involves an expansionary policy is reminiscent of Aiyagari's result that the return on the savings should be taxed. However, the model and the mechanism underlying these results are different. In our setup there is no public good through which the government can "insure" the agents. This is important because under our maintained assumption the government *does not know* who is productive, and hence it has no ability to tax productive agents to redistribute to unproductive agents.¹⁶ Another difference with Aiyagari is that the economy we consider has cycles: aggregate production, the real interest rate, and the value of money fluctuate through time. Because of these features, the method that we use to characterize the optimal policy is different.

B Proof of Remark 2

Let $i = 1$ be the index for the unproductive agent, and consider her decision problem. The money supply growth is $\mu = \frac{\dot{m}_t}{m_t}$ and let $\tilde{\xi}_t^1$ denote the lagrange multiplier of the money flow constraint in [equation \(3\)](#). The first order condition with respect to c_t^1 gives: $\xi_t^1 = q_t u'(c_t^1)$, where we used the homogeneity of degree -1 in the aggregate money supply m_t for both $\tilde{\xi}_t^1$ and \tilde{q}_t , namely: $\tilde{\xi}_t^1 = \frac{1}{m_t} \xi_t^1$ and $\tilde{q}_t = \frac{1}{m_t} q_t$. The Euler equation for \dot{m}_t^1 gives $\frac{\dot{\tilde{\xi}}_t^1}{\tilde{\xi}_t^1} = \rho$ or $\rho = \frac{\dot{\xi}_t^1}{\xi_t^1} - \frac{\dot{m}_t}{m_t} = \frac{\dot{q}_t}{q_t} - \theta \frac{\dot{c}_t^1}{c_t^1} - \mu$ where the last equality uses $\xi_t^1 = q_t u'(c_t^1)$ and that $u''c/u' = -\theta$. Notice that this is solved by $c_t^1 = \bar{c}$, $\dot{c}_t^1 = \dot{q}_t = 0$, $\mu = -\rho$ and the constant level of q is pinned down by the unproductive agent budget constraint,

$$m_0^1 + \int_0^T \dot{m}_t^1 dt = \int_0^T \frac{\bar{c}}{\bar{q}_t} dt \quad \text{which gives} \quad q = 2 \bar{c} \frac{e^{\rho T} - 1}{\rho} .$$

It is immediate to verify that this allocation also solves the Euler equation of the problem of the productive agent.

C Fiscal policy under alternative government powers

A central assumption in our analysis is that the government does not know $s(t, \omega)$, i.e. the identity of the productive type. It is useful to explore the consequences of relaxing this assumption to better understand the nature of the monetary policy problem. Without loss of generality, given the symmetry of the states, let us assume that type 1 is productive and type 2 is not productive. Also, for simplicity, let us set the money supply equal to zero in what follows.

¹⁶We discuss in [Section C](#) how the government might actually support first best through taxation if it had the ability to commit to trigger policies.

We begin by assuming that the government observes the identity of the productive type and is able to tax productive agents and transfer resources to unproductive agents. We consider two taxing technologies. The first one is lump sum taxes: in this case the productive agent pays a flat tax $\bar{\phi} = \bar{c}$, and the government uses the proceedings to finance the consumption of the unproductive agent. It is immediate that under these assumptions the complete markets allocation can be replicated. Alternatively, consider a setup where the only available taxes are distortionary, say proportional to production: then the transfer to the unproductive agent is $\tau = \phi l$. For a generic tax rate $\phi \in (0, 1)$ the consumption of the two agents solves $u'(c^1) = \frac{1}{1-\phi}$, $u'(c^2) = \frac{1}{\phi}$. An ex-ante optimal policy, maximizing the expected utility of the two types with equal weights, gives $\phi = 1/2$. Under this setting the government fiscal policy provides insurance, consumption is constant through time, though the level of consumption is smaller than under complete markets.

Table 2: Welfare (marginal utilities) under alternative fiscal policy powers

type known		type not known	
lump-sum-tax	distortionary tax	Gvt. commitment	No commitment
$u'(\bar{c}^1) = u'(\bar{c}^2) = 1$	$u'(\tilde{c}^1) = u'(\tilde{c}^2) = 2$	$u'(\bar{c}^1) = u'(\bar{c}^2) = 1$	$u'(\bar{c}^1) = 1$, $u'(c^2) = \infty$

Let us next consider a government who does not know the type's identities. In this case the efficient stationary allocation with $c = \bar{c}$ at all times for all types can be sustained if the government has the ability to commit to a trigger policy. Suppose the government credibly announces: "productive types must pay a tax \bar{c} to the government, who will then transfer it to the other types. If at any point in time the tax is not enough to pay for the transfer, the scheme will be shut down and the economy will be left in autarky forever". Assuming the threat is credible (and discounting is finite) then it is in the interest of every individual agent to comply, because deviating implies that the agent consumption is zero when unproductive which, due to Inada conditions, delivers an expected utility of $-\infty$ (since, on average, agents are unproductive for half of the times). The various outcomes sustainable under alternative fiscal policy assumptions are summarized in [Table 2](#).

In what follows, we consider a less powerful government than the one depicted above. We assume the government does not know the identity of productive types, and that it cannot commit to trigger policies. In such a situation fiscal policy, i.e. direct taxation, is powerless. Absent a liquid asset, the resource allocation is autarkic, and individuals experience inefficient fluctuations in utility. We next study the powers of monetary policy, under the maintained assumptions of type-ignorance and no-commitment.

D The Euler equation for the marginal utility of money

The maximum principle implies that the lagrange multiplier $\tilde{\gamma}$ follows¹⁷

$$\begin{aligned}
& \mathbb{E} \left\{ e^{-\rho(t+dt)} \tilde{\gamma}(m(t+dt, \omega), x(t+dt, \omega), s(t+dt, \omega)) \mid m(t, \omega) = m, x(t, \omega) = x, s(t, \omega) = 1 \right\} \\
& \approx e^{-\rho t} \left[-\rho \tilde{\gamma}(m_t, x_t, 1) dt + \tilde{\gamma}_x(m_t, x_t, 1) \dot{x}_t dt + \tilde{\gamma}_m(m_t, x_t, 1) \dot{m}_t dt + \tilde{\gamma}(m_t, x_t, 1)(1 - \lambda dt) \right] \\
& + e^{-\rho t} \tilde{\gamma}(m_t, x_t, 2) \lambda dt \\
& = \frac{e^{-\rho t}}{m_t} \left[-\rho \gamma(x_t, 1) dt + \gamma_x(x_t, 1) \dot{x}_t dt - \gamma(x_t, 1) \frac{\dot{m}_t}{m_t} dt + \gamma(x_t, 1)(1 - \lambda dt) + \gamma(x_t, 2) \lambda dt \right] \\
& = \frac{e^{-\rho t}}{m_t} \left[\gamma(x_t, 1) + \gamma_x(x_t, 1) \dot{x}_t dt - \gamma(x_t, 1)(\rho + \lambda + \mu) dt + \gamma(x_t, 2) \lambda dt \right]
\end{aligned}$$

Subtracting $e^{-\rho t} \gamma(x_t, 1)/m_t$ from both sides, dividing by dt , taking the limit for $dt \downarrow 0$, and using [equation \(12\)](#) to replace \dot{x} with its law of motion, gives the delay differential [equation \(8\)](#). An identical logic gives [equation \(9\)](#).

E Proof of [Lemma 1](#)

That $\gamma(x, s) > 0$ for $s = 1, 2$ is implied for all internal solutions from the Khun-Tucker theorem and increasing utility. Now we show that

$$\gamma(x, 1) \leq \gamma(x, 2) \quad . \quad (21)$$

Note that at any point in a history, (t, ω) , these two multipliers differ only because of the production possibility, i.e. $l^i = 0$ when the agent is unproductive ($s = 2$). Recall the first order condition with respect to c^i in [equation \(7\)](#): $\gamma(x, 2) = q(x, 2) u'(c^i(x, 2))$. Differentiating this condition with respect to l^i gives

$$\frac{\partial \gamma(x, 2)}{\partial l^i} = q(x, 2) u''(c^i(x, 2)) \frac{\partial c^i(x, 2)}{\partial l^i} = q(x, 2) u''(c^i(x, 2)) \leq 0 .$$

This shows that, at any given level of x , the multiplier is decreasing in l^i , hence $\gamma(x, 1) \leq \gamma(x, 2)$ since $l^i = 0$ when the agent is not productive. The interpretation is immediate: the multipliers $\gamma(x, 2)$ and $\gamma(x, 1)$ describe a problem that only differs with respect to the constraint on the labor supply. Since the agent has less “resources”, it follows that an additional unit of money (or “resources”) is more valuable when $s = 2$.

Next we show that $\dot{x}(x, 2) < 0$ and $\gamma_x(x, 2) < 0$ for $x \in (0, 1)$. Note that the right hand side of [equation \(9\)](#) and the inequality in [equation \(21\)](#) imply that these terms have the same sign. We now argue that the sign must be negative, otherwise the optimality of the consumption plan is violated. Differentiating the unproductive agent first order condition ([equation \(7\)](#)) gives

$$\frac{\dot{\gamma}(x, 2)}{\gamma(x, 2)} = \frac{\dot{q}(x, 2)}{q(x, 2)} + \frac{u''(c^1(x, 2))}{u'(c^1(x, 2))} \dot{c}^1(x, 2)$$

¹⁷The variable $\tilde{\gamma}$ is a costate. This condition is equivalent to the first order condition $\dot{\tilde{\gamma}} - \rho \tilde{\gamma} = 0$ in the current value Hamiltonian program.

or, using that $q(x, 2) = \gamma(1 - x, 1)$ and $\dot{c} = c_x \dot{x}$,

$$c_x^1(x, 2) \dot{x}(x, 2) = \left[\frac{\dot{\gamma}(x, 2)}{\gamma(x, 2)} - \frac{\dot{\gamma}(1 - x, 1)}{\gamma(1 - x, 1)} \right] \frac{u'(c^1(x, 2))}{u''(c^1(x, 2))}$$

Using [equation \(8\)](#) and [equation \(9\)](#) to replace the terms in the square parenthesis gives

$$c_x^1(x, 2) \dot{x}(x, 2) = \lambda \left[\frac{\gamma(1 - x, 2)}{\gamma(1 - x, 1)} - \frac{\gamma(x, 1)}{\gamma(x, 2)} \right] \frac{u'(c^1(x, 2))}{u''(c^1(x, 2))} < 0 \quad (22)$$

where the inequality follows since the term in the square parenthesis is positive, as implied by [equation \(21\)](#). The right hand side of [equation \(22\)](#) implies that consumption is decreasing, i.e. $\dot{c}^1(x, 2) < 0$. This could happen in one of two ways. First, with $\dot{x}(x, 2) > 0$ and $c_x^1(x, 2) < 0$. But this violates optimality: consumption is *decreasing* in the agent's wealth share. The agent could deviate from this plan and increase her welfare. The other possibility, consistent with optimality, is that $\dot{x}(x, 2) < 0$ and $c_x^1(x, 2) > 0$.

Finally we show that $\lim_{x \downarrow 0} \dot{x}(x, 2) = 0$. The right hand side of [equation \(22\)](#) is strictly negative at all x , including the boundary $x = 0$. Since x cannot be negative, this implies that $c_x^1(x, 2) \uparrow +\infty$ and $\dot{x}(x, 2) \uparrow 0$ as $x \downarrow 0$. [Equation \(24\)](#) in [Appendix H](#) can be used to verify that $\lim_{x \downarrow 0} c_x^1(x, 2) = +\infty$.

F Proof of [Lemma 2](#)

We will first prove by contradiction that $\bar{x} \notin [0, 1/2)$. Then we will show that $\bar{x} = 1/2$ it is enough to cover the lifetime tax obligations. Suppose that $\bar{x} < 1/2$. Without loss of generality assume that $x_t \in (\bar{x}, 1/2)$ and agent 1 is unproductive. Conditional on no reversal of the state, it follows that $x_{t+dt} < x_t$. Then for a given $\Delta \in \mathbb{R}^+$, $\Pr[x_{t+\Delta} < \bar{x}] > 0$ and therefore the agent will fail to comply with her tax obligations with positive probability. Then, $\bar{x} \notin [0, 1/2)$. Consider now the case where $x_t = \bar{x} = 1/2$. As the agent can decide not to trade she can always keep her share of outstanding money balances x above $1/2$ and therefore for any $\mu \in (0, 1)$ she will be able to cover her tax needs. That $x = 1/2$ is the ergodic set is trivial. If $x_0 < 1/2$ there is a positive probability that an agent fails to pay for her lifetime taxes. An unproductive agent with money holdings $x > 1/2$ is willing to buy goods (and the productive one with $x < 1/2$ willing to take the money) until x reaches $1/2$.

G Proof of [Proposition 2](#)

Conjecture that $c^1(x, s) = c^2(x, s) = \bar{c} \forall x, s$. From the first order conditions in [equation \(7\)](#),

$$\gamma(x, 2) = q(x, 2) = q(1 - x, 1) = \gamma(1 - x, 1) \quad \forall x$$

as $u'(\bar{c}) = 1$. When $x = 0$ we have that

$$\gamma(0, 2) = \gamma(1, 1) \equiv \gamma_a$$

and when $x = 1$ we get

$$\gamma(1, 2) = \gamma(0, 1) \equiv \gamma_b$$

There are three possibilities: (i) $\gamma_a < \gamma_b$, (ii) $\gamma_a > \gamma_b$, and (iii) $\gamma_a = \gamma_b$. In case (i) we have that $\gamma(0, 1) > \gamma(0, 2)$ which contradicts [Lemma 1](#). In case (ii) we have that $\gamma(1, 1) > \gamma(1, 2)$ which again contradicts [Lemma 1](#). The only remaining possibility is (iii) where the marginal value of money is constant for every pair $\{x, s\}$. Let $\bar{\gamma}$ denote this value. From [equation \(8\)](#), after imposing the constant value $\bar{\gamma}$ for the marginal value of money, we obtain

$$(\rho + \mu)\bar{\gamma} = 0 \tag{23}$$

which is satisfied if $\mu = -\rho$ or when $\bar{\gamma} = 0$. When $\bar{\gamma} = 0$ note that the price level has to be constant and equal to the marginal value of money $\bar{\gamma}$ which follows from direct inspection of the first order conditions presented in [equation \(7\)](#). Let \bar{q} denote this constant level. Because $\bar{\gamma} = 0$, $\bar{q} = 0$ which cannot constitute an equilibrium as one of the conditions for a monetary equilibrium is positive price \bar{q} (see [Definition 1](#)). When $\mu = -\rho$ we know from [Section 3.1](#) that it cannot constitute a monetary equilibrium. This implies that agents do not trade, money is valueless, and as a result there is no monetary equilibrium. This completes the proof of [Proposition 2](#).

This result does not hinge on μ being constant. Let $\mu = \mu(x)$. Note that [equation \(23\)](#) has to hold for every value of x . This again implies that solution requires either $\bar{\gamma} = 0$ or $\mu(x) = -\rho \quad \forall x$, which we know does not constitute an equilibrium.

H Proof of [Lemma 3](#)

That $Q_1 > 1$ follows since $\gamma(0, 2) > \gamma(0, 1)$. Using [equation \(16\)](#) we write

$$\frac{c_x^1(x, 2)}{c^1(x, 2)} = \frac{\gamma_x(1-x, 1)}{\gamma(1-x, 1)} - \frac{\gamma_x(x, 2)}{\gamma(x, 2)}$$

From [equation \(14\)](#) (evaluated at $1-x$) and [equation \(15\)](#) (evaluated at x) we get

$$\frac{\gamma_x(1-x, 1)}{\gamma(1-x, 1)} = \frac{\rho + \lambda + \mu - \lambda \frac{\gamma(1-x, 2)}{\gamma(1-x, 1)}}{\mu \left(x - \frac{1}{2}\right) + \frac{1}{\gamma(x, 2)}}, \quad \frac{\gamma_x(x, 2)}{\gamma(x, 2)} = \frac{\rho + \lambda + \mu - \lambda \frac{\gamma(x, 1)}{\gamma(x, 2)}}{\mu \left(\frac{1}{2} - x\right) - \frac{1}{\gamma(x, 2)}}.$$

Then, noting that $\gamma(0, 2) = \frac{2}{\mu}$, some algebra gives that

$$\lim_{x \downarrow 0} \frac{c_x^1(x, 2)x}{c^1(x, 2)} = 1 + \frac{\rho}{\mu} + \frac{\lambda}{\mu} \left(1 - \frac{\gamma(0, 1)}{\gamma(0, 2)}\right) \equiv Q_1 \tag{24}$$

which proves the lemma.

I Derivation of the invariant wealth distribution

The CDF for the money holdings, $F(x, s, t)$, with density $f(x, s, t)$ in states $s = 1, 2$ follows

$$\begin{aligned} F(x, 1, t + dt) &= (1 - \lambda dt)F(x - \dot{x}(x, 1) dt, 1, t) + \lambda dt F(x - \dot{x}(x, 2) dt, 2, t) \\ F(x, 2, t + dt) &= (1 - \lambda dt)F(x - \dot{x}(x, 2) dt, 2, t) + \lambda dt F(x - \dot{x}(x, 1) dt, 1, t) \end{aligned}$$

Expanding $F(x, s, t)$ around x gives (we only report the one for $s = 2$)

$$F(x, 2, t + dt) = (1 - \lambda dt) [F(x, 2, t) - f(x, 2, t) \dot{x}(x, 2) dt] + \lambda dt [F(x, 1, t) - f(x, 1, t) \dot{x}(x, 1) dt]$$

Subtracting $F(x, 2, t)$ from both sides and dividing by dt and taking the limit for $dt \downarrow 0$

$$\lim_{dt \downarrow 0} \frac{F(x, 2, t + dt) - F(x, 2, t)}{dt} = \frac{\partial F(x, 2, t)}{\partial t} = -f(x, 2, t) \dot{x}(x, 2) - \lambda (F(x, 2, t) - F(x, 1, t))$$

Using this equation together with the corresponding one for state $s = 1$ and imposing invariance give

$$0 = f(x, 2) \dot{x}(x, 2) + f(x, 1) \dot{x}(x, 1) \quad (25)$$

Taking the derivative w.r.t. x delivers the Kolmogorov forward equation

$$\frac{\partial}{\partial x} \frac{\partial F(x, 2, t)}{\partial t} = \frac{\partial f(x, 2, t)}{\partial t} = \frac{\partial [-f(x, 2, t) \dot{x}(x, 2) - \lambda (F(x, 2, t) - F(x, 1, t))]}{\partial x}$$

which, equated to zero (imposing invariance) gives [equation \(18\)](#).

I.1 Application to model with correlated shocks

Using the expressions in [equation \(12\)](#) and [equation \(13\)](#) to replace $\dot{x}(x, 1)$ and $\dot{x}(x, 2)$ into [equation \(18\)](#) gives

$$\frac{f_x(x, 2)}{f(x, 2)} = \frac{(\lambda - \mu)\gamma(x, 2) + \frac{\gamma_x(x, 2)}{\gamma(x, 2)} - \lambda\gamma(x, 2)\frac{f(x, 1)}{f(x, 2)}}{1 + \mu(x - \frac{1}{2})\gamma(x, 2)}$$

Using [equation \(25\)](#) to replace the ratio $f(x, 2)/f(x, 1)$ in the above expression gives [equation \(19\)](#).

I.2 Application to model with uncorrelated shocks

For $x \in [0, \bar{x})$ we have $f(x, 1) = 0$. Using [equation \(27\)](#) to replace $\dot{x}(x, 2)$ in [equation \(18\)](#)

$$\frac{f_x(x, 2)}{f(x, 2)} = \frac{(\lambda - \mu)\gamma(x, 2) + \frac{\gamma_x(x, 2)}{\gamma(x, 2)}}{1 + \mu(x - 1)\gamma(x, 2)}$$

Using [equation \(29\)](#) to replace for $\frac{\gamma_x(x, 2)}{\gamma(x, 2)}$ in the above expression gives [equation \(32\)](#).

J Proof of Lemma 4

See $\Omega(x)$ in [equation \(19\)](#). Because $\gamma(x, 1)$ and $\gamma(x, 2)$ are continuous and differentiable functions of x in $(0, 1)$, by inspection we know that $\Omega(x)$ is continuous and differentiable in x for any $x \in (0, 1)$.

Note that $f(x, 2) = C e^{\int_{1/2}^x \Omega(z) dz}$, which follows from integrating the ODE in [equation \(19\)](#). Because $\Omega(x)$ is continuous and differentiable for every $x \in (0, 1)$, it follows immediately that $f(x, 2)$ is continuous and differentiable for every $x \in (0, 1)$.

We now turn to show that $\lim_{x \downarrow 0} f(x, 2) = +\infty$ and $\lim_{x \downarrow 0} f_x(x, 2) = -\infty$. We will do this by looking at the behavior of a different function that is proportional to $f(x, 2)$ close to 0.

Because $\Omega(x)$ is continuous as a function of $\gamma(x, 1)$ and $\gamma(x, 2)$ there exists a positive constant k_x such that for any x

$$d \left(\Omega(x), -\frac{(\rho + \lambda + \mu)\frac{2}{\mu} - \lambda\gamma(0, 1)}{4x^2} - \frac{\lambda}{\mu(\frac{1}{2} - x) + \frac{1}{\gamma(1, 2)}} \right) < k_x, \quad \mu > 0$$

where we used that $\gamma(0, 2) = \frac{2}{\mu}$. Let $k \equiv \max k_x$, and $\hat{\Omega}(x) \equiv -\frac{(\rho + \lambda + \mu)\frac{2}{\mu} - \lambda\gamma(0, 1)}{4x^2} - \frac{\lambda}{\mu(\frac{1}{2} - x) + \frac{1}{\gamma(1, 2)}}$.

This implies that

$$d \left(\Omega(x), \hat{\Omega}(x) \right) < k \quad \forall x \in (0, 1), \quad \mu > 0$$

in other words, the distance of these two functions is uniformly bounded.

Let $\hat{f}(x, 2) \equiv \hat{C} e^{\int_{1/2}^x \hat{\Omega}(z) dz}$. Note that

$$0 < \lim_{x \downarrow 0} \frac{f(x, 2)}{\hat{f}(x, 2)} = \frac{C}{\hat{C}} e^{\lim_{x \downarrow 0} \int_{1/2}^x [\Omega(z) - \hat{\Omega}(z)] dz} < K$$

where K is a positive constant. This result follows because the distance of $\Omega(x) - \hat{\Omega}(x)$ is uniformly bounded. Then, the limiting behavior of $\hat{f}(x, 2)$ is the same as the limiting behavior of $f(x, 2)$.

Consider the expression for $\hat{\Omega}(x)$. When $x \downarrow 0$ the last term $\frac{\lambda}{\mu(\frac{1}{2} - x) + \frac{1}{\gamma(1, 2)}}$ converges to $Q_2 \equiv \frac{\lambda}{\frac{\mu}{2} + \frac{1}{\gamma(1, 2)}}$ so

$$\lim_{x \downarrow 0} \hat{\Omega}(x) = \lim_{x \downarrow 0} -\frac{Q_1}{2x^2} - Q_2$$

where $Q_1 = 1 + \frac{\rho}{\mu} + \frac{\lambda}{\mu} \left(1 - \frac{\gamma(0, 1)}{\gamma(0, 2)} \right) > 1$ as described in [Lemma 3](#).

Note that then we have the following equation for x in the neighborhood of 0,

$$\lim_{x \downarrow 0} \frac{\hat{f}_x(x, 2)}{\hat{f}(x, 2)} = \lim_{x \downarrow 0} -\frac{Q_1}{2x^2} - Q_2$$

which is an ordinary differential equation. Solving the ODE and noting that the constant of

integration is a constant $\hat{C} > 0$ gives

$$\lim_{x \downarrow 0} \hat{f}(x, 2) = \lim_{x \downarrow 0} \hat{C} e^{(\frac{Q_1}{2x} - Q_2 x)}$$

with $\lim_{x \downarrow} \hat{f}(x, 2) = +\infty$ and $\lim_{x \downarrow 0} \hat{f}_x(x, 2) = -\infty$. Therefore, $\lim_{x \downarrow} f(x, 2) = +\infty$ and $\lim_{x \downarrow 0} f_x(x, 2) = -\infty$.

K Proof of Lemma 5

First we show that a policy with $\mu \uparrow \infty$ implies that trade approaches zero. From [equation \(10\)](#) we have that $\gamma(0, 2) = 2/\mu$ and from [Lemma 1](#) that $\gamma_x(x, 2) < 0$ and $\gamma(x, 1) < \gamma(x, 2)$ for all x . As $\mu \uparrow \infty$, these imply that $\gamma(x, s) \downarrow 0$ for all x and s .

From [equation \(19\)](#) we can rewrite $f_x(x, 2)/f(x, 2)$ as

$$\begin{aligned} \frac{f_x(x, 2)}{f(x, 2)} &= \frac{\gamma(x, 2) \left(\frac{\lambda}{\mu} - 1\right)}{\frac{1}{\mu} + \left(x - \frac{1}{2}\right) \gamma(x, 2)} - \frac{\gamma(x, 2) \left(\frac{\rho + \lambda}{\mu} + 1\right)}{\mu \left(\frac{1}{\mu} + \left(x - \frac{1}{2}\right) \gamma(x, 2)\right)^2} \\ &\quad + \frac{\lambda \gamma(x, 1)}{\left[1 + \mu \left(x - \frac{1}{2}\right) \gamma(x, 2)\right]^2} - \frac{\lambda}{\mu \left(\frac{1}{2} - x\right) + \frac{1}{\gamma(1-x, 2)}} \end{aligned}$$

taking the limit as $\mu \uparrow \infty$,

$$\lim_{\mu \rightarrow \infty} \frac{f_x(x, 2)}{f(x, 2)} = \frac{1}{\frac{1}{2} - x} \text{ for } x > 0$$

which is an ODE with solution $f(x, 2) = \frac{C_0}{\frac{1}{2} - x}$. For $x > 0$ the only consistent solution is $C_0 = 0$ (otherwise $f(x, 2) < 0$ for some values of x). This implies that $f(x, 2)$ is discontinuous. Notice that $\lim_{\mu \rightarrow \infty} \frac{f_x(x, 2)}{f(x, 2)}$ is negative for $x > \frac{1}{2}$ and positive for $x < \frac{1}{2}$. In particular

$$\lim_{x \uparrow \frac{1}{2}} \left(\lim_{\mu \rightarrow \infty} \frac{f_x(x, 2)}{f(x, 2)} \right) = +\infty \quad , \quad \lim_{x \downarrow \frac{1}{2}} \left(\lim_{\mu \rightarrow \infty} \frac{f_x(x, 2)}{f(x, 2)} \right) = -\infty$$

which tells us that there is a mass point at $\frac{1}{2}$. Also note that $\lim_{\mu \rightarrow \infty} \frac{f_x(0, 2)}{f(0, 2)} = 0$ so there cannot be a mass point at $x = 0$. Then, there exists a unique mass point at $x = \frac{1}{2}$,

$$f(x, 2) = \begin{cases} 1 & \text{for } x = 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Because $f(1/2, 2) = f(1/2, 1) = 1$, a productive agent has no incentives to sell goods to an unproductive agent because she cannot affect the ex-post distribution of money holdings, and therefore the economy approaches autarky.

Recall that [Proposition 1](#) showed that any policy with $\mu < 0$ also implies that in the ergodic set the economy is in autarky.

Hence, the expected utility $\mathcal{W}(\mu)$ diverges to $-\infty$ for both of these cases: the reason is that ex-ante agents are unproductive 1/2 of the times, and in this state their consumption is zero.

L Uncorrelated productivity shocks (Bewley economy)

In this section we study an economy where the productivity shocks are uncorrelated across agents. Assume a unit mass of agents, indexed by i , over the $[0, 1]$ interval. As before, the productivity state of each agent, s_i follows a Markov process, where λ denotes the rate at which the state switches.

Let m_t^i be the money holdings of agent i at time t , so that the total money supply is $m_t = \int_0^1 m_t^i di$. Let τ_t denote the per capita transfer from the government. The government budget constraint is

$$\tau_t = \tilde{q}_t \int_0^1 \dot{m}_t^i di = \mu m_t \tilde{q}_t = \mu q_t \quad (26)$$

where the last equality uses the homogeneity of \tilde{q} with respect to m . In what follows we let $\mu \geq 0$. The same argument developed in [Section 3.1](#) shows that no monetary equilibrium exists for $\mu < 0$.

Obviously $\int_0^1 x_t^i di = 1$ where $x_t^i = m_t^i/m_t$. Notice that in this model with a continuum of agents $x_i \in [0, +\infty)$, where $x_i = 1$ denotes the situation in which a single agent money balances equal the economy's average money, and $x_i \uparrow \infty$ denotes the situation in which one agent holds all of the money. Simple algebra gives that $\dot{x}(x, s) = \dot{m}_t^i/m_t - x\mu$, where we omit the time and agent i indices for notation simplicity.

The agent's first order conditions of this model are unchanged compared to the previous model ([equation \(7\)](#)). The unproductive agent budget constraint and Euler equation (i.e. when $s = 2$) give

$$\dot{x}(x, 2) = \mu(1 - x) - \frac{1}{\gamma(x, 2)} \quad (27)$$

This equation shows that money growth has no effect on the money share in the case where the agent's money holding equal the average money per capita in the economy, i.e. the ratio is $x_i = 1$.¹⁸

We assume the economy has a centralized competitive market where one unit of money buys $\tilde{q} = \frac{1}{m}q$ units of consumption. This implies that all productive agents are willing to produce in exchange for money as long as $\gamma(x, 1) > q$, and that there is a level of money holdings \bar{x} where productive agents are satiated with money balances: $\gamma(\bar{x}, 1) = q$.

Every agent works to save money balances \bar{x} as soon as he gets productive. So the wealth share jumps from x to \bar{x} as soon as the agent become productive (this implies that \dot{x} is infinite at the time of a jump).¹⁹ A productive agent aims to maintain the wealth share constant at x so that $\dot{x}(\bar{x}) = 0$, which implies that for $x = \bar{x}$ then $\dot{m}_t^i/m_t = \mu\bar{x}$.

¹⁸Note that the same property holds in the model with two agents, in which the total mass is 2, the index $x^i \in [0, 1]$ and an equal distribution of money holdings implies that the ratio of the agent money to the average (per capita) money holdings is 1/2, so that μ does not affect \dot{x} when $x = 1/2$.

¹⁹The assumption of linear disutility of labor is important for this result, as it implies that the productive agent immediately refills his money balances up to \bar{x} .

Before moving on with solving the model we define an equilibrium for this economy.

Definition 2 *A monetary equilibrium is a price function $\tilde{q}(m) = \frac{1}{m} q$, with $q \in \mathbb{R}^+$, and a stochastic process $x(t, \omega)$ with values in $[0, 1]$, such that a consumer i maximizes expected discounted utility (equation (1)) subject to the constraints (equations (2), (3), and (4)) with $q(t, \omega) = q(s)$ and the government budget constraint (equation (26)) is balanced*
ADD TAX SOLVENCY.

In an internal solution the lagrange multipliers $\gamma(x, 1), \gamma(x, 2)$ solve the system of differential equations that we determined before (equation (8) and (9)), which under the assumptions of this section (using equation (27)) gives

$$\begin{aligned} \gamma_x(x, 1)\dot{x}(x, 1) &= (\rho + \mu)\gamma(x, 1) - \lambda(\gamma(\bar{x}, 2) - \gamma(\bar{x}, 1)) \\ \gamma_x(x, 2) \left[\mu(1-x) - \frac{1}{\gamma(x, 2)} \right] &= (\rho + \mu)\gamma(x, 2) - \lambda(\gamma(x, 1) - \gamma(x, 2)) \end{aligned}$$

The system decouples in two ODEs which we discuss next.

Internal solution always applies for $x \in (0, 1)$ for the unproductive agent. For the productive agent the solution is internal only at $x = \bar{x}$, otherwise the state x records a jump from x to \bar{x} . At the replenishment level \bar{x} the equation for the productive agent gives

$$q = \gamma(\bar{x}, 1) = \frac{\lambda}{\lambda + \mu + \rho} \gamma(\bar{x}, 2) \quad (28)$$

Since the disutility of labor is linear, the marginal utility of money for a productive agent is constant at the level $\gamma(\bar{x}, 1) = \gamma(x, 1)$ for $x \in (0, \bar{x})$.²⁰ The ODE for $\gamma(x, 2)$ can be rewritten as

$$\gamma_x(x, 2) = \frac{(\rho + \lambda + \mu)\gamma(x, 2)^2 - \lambda q \gamma(x, 2)}{\gamma(x, 2)\mu(1-x) - 1} \quad (29)$$

with the following boundary conditions at $x = 0$:

$$\gamma(0, 2) = \frac{1}{\mu} \quad (30)$$

where the boundary stems from the unproductive agent's Euler equation and budget constraint at $x = 0$.

Next we impose that at every t demand equals supply in the asset market where money is exchanged for the consumption good. The asset demand originates from the productive agents who aim to hold an amount of money \bar{x} . The supply comes from unproductive agents who exchange money for consumption. Let $f(x, 2)$ be the density function for the mass of unproductive agents with $\int_0^{\bar{x}} f(x, 2) dx = 1$, so that $f(x, 2)$ is the measure of unproductive agent, who account for 1/2 of the total population under the invariant. The market clearing equation is (see below)

$$(\mu + 2\lambda)(\bar{x} - 1) = \int_0^{\bar{x}} \frac{f(x, 2)}{\gamma(x, 2)} dx \quad (31)$$

²⁰Hence the value (i.e. the utility level) for an unproductive agent with $x \in (0, \bar{x})$ is linear in x .

Finally note that the density function for unproductive agents is obtained from the usual Kolmogorov forward equation. Using $f(x, 1) = 0$ and [equation \(27\)](#) to replace $\dot{x}(x, 2)$ into [equation \(18\)](#) gives

$$\frac{f_x(x, 2)}{f(x, 2)} = \frac{(\lambda - \mu)\gamma(x, 2) - \frac{(\rho + \lambda + \mu)\gamma(x, 2) - \lambda q}{1 + \mu(x-1)\gamma(x, 2)}}{1 + \mu(x-1)\gamma(x, 2)} \quad (32)$$

Notice that this problem has four unknowns: q , \bar{x} , and two constant of integration, one for the solution of [equation \(29\)](#) and one for the density function in [equation \(32\)](#). The four equations to solve for the four unknowns are: the boundary conditions in [equation \(28\)](#) and [equation \(30\)](#), the market clearing condition in [equation \(31\)](#), and the density function integrating to a unit mass.

Derivation of market clearing equation

We derive [equation \(31\)](#) as the limit of a discrete time model. Recall that each productive agent is aiming to keep his money balances at $m^i/m = \bar{x}$. Consider a time interval of length Δ . In each period there is a fraction $\Delta\lambda$ of the mass of unproductive agents who become productive. The asset demand for one of these agents with wealth x is given by a change in the stock, $\bar{x} - x$, and by a flow component that offsets the effect of money growth given by $\mu\Delta(\bar{x} - 1)$. The complementary fraction of productive agents, $1 - \lambda\Delta$, is formed by agents who were already productive in the previous period and held assets \bar{x} . The asset demand for these agents is the flow component that offsets the effect of money growth, given by $\mu\Delta(\bar{x} - 1)$. This gives

$$(1 - \Delta\lambda)\mu\Delta(\bar{x} - 1) + \lambda\Delta \int_0^{\bar{x}} [\bar{x} - x + \mu\Delta(\bar{x} - 1)] f(x, 2) dx = \\ (1 - \Delta\lambda) \int_0^{\bar{x}} \frac{c^1(x, 2)\Delta}{q} f(x, 2) dx + \lambda\Delta \frac{c^1(\bar{x}, 2)\Delta}{q}$$

Dividing by Δ and taking the limit as $\Delta \downarrow 0$ gives

$$\mu(\bar{x} - 1) + \lambda \int_0^{\bar{x}} (\bar{x} - x) f(x, 2) dx = \int_0^{\bar{x}} \frac{c^1(x, 2)}{q} f(x, 2) dx$$

Using $\int_0^{\bar{x}} f(x, 2) dx = 1$ and $c^1(x, 2) = q/\gamma(x, 2)$ gives

$$\mu(\bar{x} - 1) + \lambda\bar{x} - \lambda \int_0^{\bar{x}} x f(x, 2) dx = \int_0^{\bar{x}} \frac{1}{\gamma(x, 2)} f(x, 2) dx$$

Using that $\int_i x^i di = 1$ or, equivalently, $\frac{1}{2} \int_0^{\bar{x}} x f(x, 2) dx + \frac{\bar{x}}{2} = 1$ gives

$$\mu(\bar{x} - 1) + \lambda\bar{x} - \lambda(2 - \bar{x}) = \int_0^{\bar{x}} \frac{1}{\gamma(x, 2)} f(x, 2) dx$$

or

$$\mu(\bar{x} - 1) + 2\lambda(\bar{x} - 1) = \int_0^{\bar{x}} \frac{1}{\gamma(x, 2)} f(x, 2) dx$$

which gives [equation \(31\)](#).

L.1 The case with constant money ($\mu = 0$)

It is interesting to analyze the case with $\mu = 0$ for the model with uncorrelated shocks because of its simplicity. We provide a characterization of the equilibrium and we prove existence.

After setting $\mu = 0$ [equation \(29\)](#) reduces to the following ODE: $\gamma_x(x, 2) = \lambda q \gamma(x, 2) - (\rho + \lambda) \gamma(x, 2)^2$, with boundary conditions that reduce to $\lim_{x \downarrow 0} \gamma(x, 2) = \infty$ and $\gamma(\bar{x}, 2) = q \frac{\rho + \lambda}{\lambda}$. Using the ODE and the first boundary provides an expression for the marginal value of money for an unproductive agent as a function of the price level q ,

$$\gamma(x, 2) = \frac{q\lambda}{\rho + \lambda} \frac{e^{q\lambda x}}{e^{q\lambda x} - 1}$$

which is strictly positive for every $q > 0$. This implies that the expected utility for an unproductive agent with wealth x , denoted by $v(x, 2)$, is given by the integral of $\gamma(x, 2)$, i.e.

$$v(x, 2) = \frac{1}{\rho + \lambda} \log(e^{q\lambda x} - 1) + \bar{v}$$

where \bar{v} is a finite constant.

Using the expression for $\gamma(x, 2)$ and [equation \(28\)](#) gives an equation that relates \bar{x} and q ,

$$\bar{x} = \frac{1}{\lambda q} \log \left(\frac{(\lambda + \rho)^2}{(\lambda + \rho)^2 - \lambda^2} \right) \quad (33)$$

which implies a unit elasticity of \bar{x} with respect to the price level q .

Now we turn to evaluate the density of money holdings $f(x, 2)$. Setting $\mu = 0$ in [equation \(32\)](#) reduces to

$$\frac{f_x(x, 2)}{f(x, 2)} = \lambda q \left(1 - \frac{\rho}{\lambda + \rho} \frac{e^{q\lambda x}}{e^{q\lambda x} - 1} \right)$$

where we used the expression we found for $\gamma(x, 2)$. This expression is an ODE with solution $f(x, 2) = Q_7 e^{q\lambda x} (e^{q\lambda x} - 1)^{-\frac{\rho}{\rho + \lambda}}$, where Q_7 is a constant to be determined next. We have that

$$\int_0^{\bar{x}} f(y, 2) dy = 1$$

We can use this equation to obtain Q_7 . Therefore, the density $f(x, 2)$ is

$$f(x, 2) = \frac{q\lambda^2}{(\rho + \lambda) (e^{q\lambda \bar{x}} - 1)^{\frac{\lambda}{\rho + \lambda}}} \frac{e^{q\lambda x}}{(e^{q\lambda x} - 1)^{\frac{\rho}{\rho + \lambda}}}$$

which is readily evaluated using [equation \(33\)](#).

An equilibrium exists if there exists a finite price level q such that the market clearing condition (see [equation \(31\)](#)) is satisfied. Using that $\mu = 0$ and by substituting the expressions we found for the marginal value of money $\gamma(x, 2)$, density function $f(x, 2)$, and that $\bar{x} = \bar{x}(q)$, the market clearing condition reduces to

$$\Upsilon_1(q) = \Upsilon_2(q)$$

where $\Upsilon_1(q) \equiv (\bar{x}(q) - 1) 2 \left(\frac{\lambda^2}{(\rho + \lambda)^2 - \lambda^2} \right)^{\frac{\lambda}{\rho + \lambda}}$ and $\Upsilon_2(q) \equiv \int_0^{\bar{x}(q)} (e^{q\lambda x} - 1)^{\frac{\lambda}{\rho + \lambda}} dx$, where both $\Upsilon_1(q)$ and $\Upsilon_2(q)$ are continuous and differentiable functions on q . Note that $\lim_{q \downarrow 0} \Upsilon_1(q) = +\infty$, $\lim_{q \uparrow \infty} \Upsilon_1(q) < 0$, $\lim_{q \downarrow 0} \Upsilon_2(q) = -\infty$ (**CHECK**), and $\lim_{q \uparrow \infty} \Upsilon_2(q) = 0$. Therefore, there exists at least one value q , with $0 < q < \infty$, such that the market clearing condition is satisfied. Therefore, we proved existence of equilibrium.

Online Appendices

The Optimum Quantity of Money with Borrowing Constraints

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A-13 The Belman approach to the optimal level of money

Let $V^1(x)$ and $V^2(x)$ denote the value functions of agents of type 1 and 2 with money holdings x . Using that from the budget constraint of a productive agent we get $\dot{x}(x, 1) = \frac{l^1 - c^1}{q(x, 1)} + \mu \left(\frac{1}{2} - x\right)$, and that $\dot{x}(x, 2) = -\frac{c^1}{q(1-x, 1)} + \mu \left(\frac{1}{2} - x\right)$, we can write the problem of both types of agents in its Belman form,

$$\begin{aligned}\rho V^1(x) &= \max_{c^1, l^1} \ln c^1 - l^1 + V_x^1(x) \left(\frac{l^1 - c^1}{q(x, 1)} + \mu \left(\frac{1}{2} - x \right) \right) + \lambda (V^2(x) - V^1(x)) \\ \rho V^2(x) &= \max_{c^2} \ln c^2 + V_x^2(x) \left(-\frac{c^1}{q(1-x, 1)} + \mu \left(\frac{1}{2} - x \right) \right) + \lambda (V^1(x) - V^2(x))\end{aligned}$$

We can obtain the derivatives of the value functions with respect to the money growth rate parameter μ by applying the envelope theorem,

$$\begin{aligned}V_\mu^1(x)(\rho + \lambda) &= V_x^1(x) \left(\frac{1}{2} - x \right) + \lambda V_\mu^2(x) \\ V_\mu^2(x)(\rho + \lambda) &= V_x^2(x) \left(\frac{1}{2} - x \right) + \lambda V_\mu^1(x)\end{aligned}$$

which can be solved to obtain

$$\begin{aligned}V_\mu^1(x) &= \left(\frac{\frac{1}{2} - x}{\rho} \right) \frac{(\rho + \lambda)V_x^1(x) + \lambda V_x^2(x)}{\rho + 2\lambda} \\ V_\mu^2(x) &= \left(\frac{\frac{1}{2} - x}{\rho} \right) \frac{(\rho + \lambda)V_x^2(x) + \lambda V_x^1(x)}{\rho + 2\lambda}\end{aligned}$$

taking the ratio,

$$\frac{V_\mu^2(x)}{V_\mu^1(x)} = \frac{(\rho + \lambda)V_x^2(x) + \lambda V_x^1(x)}{(\rho + \lambda)V_x^1(x) + \lambda V_x^2(x)} > 1$$

as $V_x^1(x) = \gamma(x, 1)$, $V_x^2(x) = \gamma(x, 2)$, and $\gamma(x, 2) > \gamma(x, 1)$ from **Lemma 1**.

The social planner wishes to maximize a weighted average of the discounted utilities,

$$V = \max_\mu \int_0^1 (\theta V^1(1-x) + (1-\theta)V^2(x)) f(x, 2) dx$$

where $\theta \in (0, 1)$ and $f(x, 2)$ is the stationary density of money holdings. The first order condition of the social planner's problem is

$$\frac{\partial V}{\partial \mu} = \int_0^1 (\theta V^1(1-x) + (1-\theta)V^2(x)) f_\mu(x, 2) dx + \int_0^1 (\theta V_\mu^1(1-x) + (1-\theta)V_\mu^2(x)) f(x, 2) dx$$

Let $\hat{\mu}$ be the optimal level of money growth rate. Then, $\hat{\mu} = 0$ if $\frac{\partial V}{\partial \mu} < 0 \forall \mu > 0$, $\hat{\mu} \uparrow \infty$ if $\frac{\partial V}{\partial \mu} > 0 \forall \mu > 0$, and $0 < \hat{\mu} < \infty$ if $\frac{\partial V}{\partial \mu} = 0$ for some $\mu > 0$. Also, as there might be local maxima, one way to know that the optimal money growth rate is not 0 is to check that

$\frac{\partial V}{\partial \mu}$ at 0 is positive.