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Time-varying (S, s) band models: empirical properties and interpretation*

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Abstract

A recent strand of empirical work uses (S, s) models with time-varying stochastic bands to describe infrequent adjustments of prices and other variables. The present paper examines some properties of this model, which encompasses most micro-founded adjustment rules rationalizing infrequent changes. We illustrate that this model is also flexible enough to fit data characterized by infrequent adjustment and variable adjustment size. We show that, to the extent that there is variability in the size of adjustments (e.g. if both small and large price changes are observed), i) a large band parameter is needed to fit the data and ii) the average band of inaction underlying the model may differ strikingly from the typical observed size of adjustment. The paper thus provides a rationalization of a recurrent empirical result: very large estimated values for the parameters measuring the band of inaction.

Keywords: (S, s) models, adjustment costs, menu costs.

JEL Codes: E31, D43, L11

Résumé

Les modèles dits (S, s) décrivent l'ajustement de variables économiques, telles que les prix ou les biens durables, comme un phénomène discret. Dans le cas des prix par exemple, la stratégie optimale du producteur est de ne modifier son prix que si l'écart entre le prix existant et le prix désiré dépasse un certain seuil. Les valeurs s et S caractérisant les seuils de baisse et de hausse des prix définissent une "bande d'inaction".

Un champ récent de la littérature empirique utilise des modèles (S, s) autorisant la bande d'inaction à varier dans le temps. Le présent article étudie quelques propriétés de ce modèle. Tout d'abord, nous montrons comment ce modèle est lié aux modèles microéconomiques structurels à coût d'ajustement, comme les coûts de menu, dans le cas où ces coûts sont aléatoires. Nous illustrons également que ce modèle est suffisamment flexible pour permettre de rendre compte simultanément de deux caractéristiques empiriques des changements de prix au niveau individuel: i) les changements de prix sont rares et ii) ils sont d'ampleur très variable. Nous montrons enfin que, dès lors qu'il existe une variabilité significative dans l'ampleur des ajustements (par exemple, si à la fois des petites et des grandes variations de prix sont observées), i) seule une valeur élevée du paramètre décrivant la bande d'inaction permet d'ajuster les données et ii) la moyenne de la bande d'inaction du modèle peut alors différer fortement de la taille moyenne des ajustements observés. Notre résultat rend ainsi compte d'un résultat empirique récurrent : de très grandes valeurs estimées pour les paramètres associés à la bande d'inaction.

Mots-clé : modèles (S, s), coûts d'ajustement, coûts de menu.

Codes JEL : E31, D43, L11

Non-technical summary

Many economic decisions are reduced, delayed or protracted due to the existence of adjustments costs. For example, retailers do not change prices of all their products every day because they have to reprint price tags or catalogues. Adjustment costs are also significant in investment, durable consumption or labor demand decisions and changes in these variables are often described as discrete processes. For instance, the price-setting policy of a producer could be summarized as resulting of a trade-off between paying the menu-cost and letting its price unchanged. The observed price is then different from the price that would be set without frictions. In the presence of non-convex adjustment costs, standard theoretical results show that the optimal strategy for the producer is to follow an (S, s) policy. The producer does not change its prices until the price gap (which is the difference between the observed price and the price that would be set without friction) exceeds a certain threshold (S or s). The thresholds s and S trigger increases and decreases and if the price gap is between these two values, it is optimal to do nothing. The band of inaction defined by these two thresholds is fully related to the size of the adjustment cost. Typically, the larger adjustment costs are, the wider the inaction band is.

However, one of the prediction of this model is that all adjustments have the same size, which is at variance with a lot of micro findings. Thus, a recent strand of the literature uses (S, s) models with time-varying bands, which help to predict that the size of adjustment can vary over time for a given firm. This article studies some statistical properties of this model.

First, using a simple stochastic menu-cost model, we obtain that, in presence of time-varying menu costs, the band of inaction is also varying over time. We still obtain a positive link between the size of the band and the size of the adjustment cost. We also illustrate that this model is flexible enough to reproduce two important micro-findings on prices: price changes are infrequent and the size of price changes varies a lot for a given firm.

Then, using a simplified model, we derive analytically some properties of the (S, s) models. In particular we show how the data moments generated by the model are related to the structural parameters of the model. Using more realistic models, we then provide simulation evidence which confirm these analytical results. Both results show that to reproduce a significant degree of variability in the size of adjustments, the band of inaction should be very large and the average size of this band $(S - s)$ is then much larger than the average size of the adjustments. In the traditional fixed band of inaction model, the size of the band is equal to the adjustment size.

These analytical and simulation results are finally confronted to empirical results obtained in the microeconomic literature on price rigidity and we provide a possible interpretation of these results.

1 Introduction

A traditional theoretical result for a number of dynamic economic problems is that in the presence of non-convex adjustment costs, the optimal decision rule has, under some specific assumptions, the form of an (S, s) rule. The (S, s) model is characterized by the existence of a "band of inaction", i.e. a range of values of the state variable for which it is found optimal not to adjust. In the case of price-setting, Sheshinski and Weiss (1977) have shown that it is optimal for firms to tolerate some deviation of their current price from their optimal frictionless price as long as this deviation is not too large. The size of the band is then an increasing function of the fixed cost of adjusting prices. The (S, s) rule has been shown to be optimal for various other economic decisions: Scarf (1959) for inventories and Grossman and Laroque (1991) for consumption and investment problems.

In the core (S, s) model, the adjustment cost is constant over time and across firms and the band of inaction is fixed. The size of observed changes is then predicted to be equal to the size of the band of inaction. In particular, for a given item, all adjustments should have a similar magnitude and be rather large. This prediction is however at variance with patterns often observed in microeconomic data. For example, Hall and Rust (2000) report a high degree of variability in investment adjustment decisions, which is highly difficult to match with a fixed band of inaction model. In the case of prices, the prevalence of small price changes and the significant variance in the size of microeconomic price adjustments has been widely documented, see inter alia Dhyne *et al.* (2006) and Klenow and Kryvstov (2008).

To reproduce volatility in the size of adjustments, a growing strand of research has proposed empirical models allowing for time-varying random (S, s) bands. Time-varying random (S, s) bands can, as discussed by Caballero and Engel (1999) and Hall and Rust (2000), be rationalized by models in which the fixed cost of adjustment is random. Stochastic adjustment costs then give rise to sizes of adjustment that vary over time for a given firm. Caballero and Engel (1999) have used such a model with Gamma-distributed adjustment costs to explain investment dynamics. Caballero, Engel and Haltiwanger (1997) used also an (S, s) model with time-varying bands to rationalize microeconomic employment adjustment policies that vary over time. Recently, Fisher and Konieczny (1995, 2006) and Dhyne, Fuss, Pesaran and Sevestre (2007) estimate (S, s) models with random thresholds using individual price data for several categories of products.[1] A recurrent finding in this literature is the rather large size of the estimated (S, s) band, typically much larger than the average observed price change. For example, Attanasio (2000) estimates time-varying (S, s) band model on durable goods data, and reports that the average size of the band is wider than the average size of price changes. He concludes that "the most striking feature, however, is the width of the band". Dhyne *et al.* (2007) also underline this result, which stands in contrast with the deterministic version of the model where the size of price change is expected to be equal to the size of the inaction band.

The present paper examines some properties of empirical random band (S, s) models, and proposes a rationalization for the above-mentionned empirical result. Using both a simple, analytically tractable, framework and simulations of a more elaborate model, we exhibit some relationships between the mean and variance of the (S, s) band and different moments of adjustments generated by the model. In particular, we show that introducing variability in the adjustment threshold increases the variance of adjustment size and at the same time reduces the average size of adjustment. Since the average size of adjustment is itself an increasing function of the bandwidth, it turns out that fitting data with substantial variability in size of adjustment requires both the mean and the variance of the (S, s) band to have large values.

The paper mostly focuses on application to price-setting, since theoretical and empirical research using time varying (S, s) band model has been growing over recent years. In particular, Dotsey, King and Wolman (1999) and Klenow and Krystov (2008) underline stochastic menu cost models are, unlike fixed menu cost models, able to rationalize the prevalence of small price changes found in micro data. Our result remain however valid for applications to other economic decisions.

This paper is structured as follows. Section 2 provides an economic motivation by considering a structural menu cost model with random adjustment cost, which gives rise to random (S, s) bands. Within a simple framework, Section 3 establishes analytically some results on the

[1] Sheshinski, Tishler and Weiss (1981) and Dahlby (1992) are early examples of contributions on price-setting relying on similar specifications.

relationship between the variance and the mean of the band and moments generated by the model (hazard rate, mean and variance of adjustments). Using more realistic models Section 4 provides simulation evidence which confirm analytical results, while Section 5 illustrates our results using actual estimates from the literature. Section 6 summarizes and draws implications for interpreting empirical evidence.

2 Time-varying (S, s) band models: a structural motivation

In this section we provide a structural motivation to the random (S, s) band model. We show that this model is able to describe the optimal microeconomic policy rule when menu costs are stochastic. In the case of price adjustment rules, Sheshinski and Weiss (1977) show that in a presence of a fixed menu cost and constant inflation, (S, s) policies are optimal. As obtained by Caballero and Engel (1999) and Hall and Rust (2007), if menu costs are randomly distributed, the optimal policy rules can be represented by models with stochastic (S, s) bands. To our knowledge however, the optimality of these generalized forms of the (S, s) policy has not been proved analytically in a general case (see Caballero and Engel (1999) for a discussion). Here, we use simulation of a calibrated model with stochastic menu-costs, solved numerically. We consider price adjustment rules and rely on a menu cost model comparable to those analyzed recently by Dotsey, King and Wolman (1999), Golosov and Lucas (2008), Klenow and Krystov (2008), and Nakamura and Steinsson (2008).

More precisely, we use Nakamura and Steisson's (2008) set-up, extending this model by introducing a stochastic rather than deterministic menu cost. This model considers, in a partial equilibrium context, the pricing decision of a firm that operates in a monopolistic competition environment. The demand adressed to the firm is $Y_t^d = D\left(\frac{P_t}{\overline{P}_t}\right)^{-\theta}$, where D is constant, P_t is the firm's price and $\overline{P}_t$ is the overall price level. The production function of the firm is linear: $Y_t = A_t N_t$ where A_t is the level of productivity and N_t total hours worked. The logarithm of productivity is assumed to follow an AR(1) process: $a_t = \rho a_{t-1} + \varepsilon_t$ where $a_t = ln(A_t)$ and $\sigma_\varepsilon^2 = E\varepsilon_t^2$. The overall price level is assumed to follow a random walk with drift $ln\overline{P}_t = \mu + \ln(\overline{P}_{t-1}) + \varepsilon_t^P$. As Nakamura and Steinsson (2008), we assume that the real wage is constant, and equal to its equilibrium level under flexible price, $\frac{W_t}{P_t} = \frac{\theta-1}{\theta}$. The period real profit function is then given by:

$$\Pi(P_t/\overline{P}_t, a_t) = \frac{P_t}{\overline{P}_t} D \left(\frac{P_t}{\overline{P}_t}\right)^{-\theta} - \left(\frac{\theta-1}{\theta}\right) \frac{Y_t}{\exp(a_t)}$$

When changing its price, the firm incurs a menu cost c_t, expressed as a fraction of steady state output. We here assume that c_t is stochastic, and drawn from a Beta distribution (following Dotsey, King and Wolman (1999)). This specification generates positive and bounded menu costs, but still allows for a wide range of cases, including as specific cases: a fixed menu cost and a bimodal distribution, which mimicks the Calvo process. The vector of state variables is $\{P_{t-1}/\overline{P}_t, a_t, , c_t\}$. At time t, assuming a discount factor of β,the value of the firm (the present value of profits) is given by:

$$V(P_{t-1}/\overline{P}_t, a_t, c_t) = \max[V^{nc}(P_{t-1}/\overline{P}_t, a_t, c_t), V^{c}(P_{t-1}/\overline{P}_t, a_t, c_t)]$$

where $V^c(P_{t-1}/\overline{P}_t, a_t, c_t)$ is the value if the firm change its prices and $V^{nc}(P_{t-1}/\overline{P}_t, a_t, c_t)$ is the value if the firm does not change its price. These two functions are given by:

$$V^c(P_{t-1}/\overline{P}_t, a_t, c_t) = \max_{p_t}[\Pi(P_t/\overline{P}_t, a_t) + \beta E_t V(P_t/\overline{P}_{t+1}, a_{t+1}, c_{t+1})] - c_t D$$

and

$$V^{nc}(P_{t-1}/\overline{P}_t, a_t, c_t) = \Pi(P_{t-1}/\overline{P}_t, a_t) + \beta E_t V(P_t/\overline{P}_{t+1}, a_{t+1}, c_{t+1})$$

To solve the model, we use a value-function iteration technique. For this purpose the processes for productivity, inflation and menu costs are discretized. We employ the Tauchen (1986) procedure to discretize productivity and inflation, while discretization of c_t is straightforward given independence across draws. We then solve the program of the firm and are able to derive the policy function and simulate the model.

We calibrate the model as follows, mainly following Nakamura and Steinsson (2008). We set the discount factor to $\beta = 0.96^{1/12}$, and the elasticity of demand to $\theta = 4$. Both values fall in standard ranges, and the latter is consistent with a mark-up of 1.33. Here, we focus on illustrating the consequences of random menu costs. So, we abstract from productivity shocks and set $\sigma_\varepsilon = 0$ and $\rho = 0$ in the simulations below. The mean of the process for overall inflation is set to $\mu = 0.002$. This value is consistent with monthly inflation rate in services in France over recent years. We also set $\sigma_\varepsilon^P = 0.02$. The menu costs are assumed to be independently drawn from a Beta distribution with mean $\mu = 0.06$, and $\sigma_\varepsilon = 0.03$.[2] Note that these moments characterize the population distribution of menu costs, but may not characterize the empirical distribution of menu costs actually paid by firms since firms will not change price independently of the realized value of the menu cost. Using this calibration, we are able to simulate a model of price adjustment for $1,000,000$ periods and we obtain a monthly frequency of price change of 6.8% (4.7% for increases and 2.1% for decreases), 30% of price decreases and an average size of absolute price changes of 6.4%. These moments are consistent with the results obtained for consumer prices in services in France (Baudry *et al.*, 2007). The frequency of price changes in services is 7.2% (5.8% for increases and 1.4% for decreases), 20% of price decreases and an average of absolute price changes around 6.5%.

To represent the pricing policy of the firm and relate this model to empirical (S, s) models, a relevant variable is the price gap, introduced in particular by Caballero and Engel (1999). If prices were flexible the nominal optimal price would be $P_t^* = (\frac{\theta}{\theta-1})\frac{W_t}{A_t}$. Hence under the above assumption the log-optimal price is $p_t^* = \overline{p}_t - a_t$. We define the price gap of a firm at date t as $z_t = p_{t-\tau} - p_t^*$, where τ is the duration elapsed since the last price change. In our case where $a_t = 0$, the policy function can here be expressed as a function of z_t and c_t following Caballero and Engel's (1999) approach. Indeed the argument of the value function is $P_{t-1}/\overline{P}_t$. The policy function is pictured in Figure 1. The realization of the menu cost is on the x-axis and the level of the pre-adjustment price gap z_t is on the y-axis. For each value of menu cost, the solid line gives the threshold values for which the firm is indifferent between changing in price and keep its price unchanged. Inside the region drawn by the curve, the price is kept unchanged whereas outside this region, the price is changed. The line describes the (S, s) band obtained for each value of the menu cost. Figure 1 illustrates that the inaction band is varying with the value of the menu cost: for larger values of menu costs, the band is larger. This result is in accordance with the inaction band obtained by Caballero and Engel (1999) for investment.

This exercise provides a structural motivation to the empirical model we study hereafter.

[2] The corresponding parameters (a, b) of the Beta distribution are $a = 3.7$ and $b = 58.0$.

Note that we here focus on the variation of menu-costs but other shocks in the model could generate time-variation in (S, s) bands. For instance Golosov and Lucas (2008) show that productivity shocks can lead to varying adjustment bands.

3 Properties of a time-varying (S, s) band models: some analytical results

In this section, using a simple model, we derive analytically some properties of the (S, s) models. In particular we show how the data moments generated by the model are related to the parameters of the model. We focus here on price-setting decisions but our results can obviously be extended to other types of economic decisions.

3.1 A simple model

We note respectively p_t and p_t^* the logarithm of the price posted by the firm at date t and the optimal price of the firm at that date. More precisely, p_t^* is the price it chooses to implement if it reprices at date t. For simplicity, we consider a model that only involves price increases. We assume that the policy of the firm is to follow a one-sided (S, s) rule, and that the gap $p_t^* - p_t$ fully describes the environment of the agent. We note S_t the time-varying threshold for price increases. That is, the firm's policy is to maintain its price unchanged as long as $p_t^* - p_{t-\tau} < S_t$ and to change its price to p_t^* whenever $p_t^* - p_{t-\tau} \geqslant S_t$.

To obtain analytical results, we make the following assumptions on the processes for p_t^* and S_t in this section. First, the optimal price follows a deterministic trend $p_t^* = \gamma t$ where $\gamma > 0$. Without restriction, we set the initial nominal price to be $p_0 = 0$ and the initial optimal price to be $p_0^* = 0$. Second, the price threshold follows a Bernoulli distribution: $S_t = S + \nu_t$ where $P(\nu_t = a) = P(\nu_t = -a) = \frac{1}{2}$. Thus, at each date the firm can either face, with an equal probability, a "low" threshold or a "high" threshold.

The model is summarized in Figure 2: the price deviation p_t^* grows until it hits the random threshold S_t. Observe that here the price can only be modified between dates $t = T^-$ and $t = T^+$ characterized by $p_{T^-}^* = \gamma T^- = S - a$ and $p_{T^+}^* = \gamma T^+ = S + a$. Thus $T^- = \frac{S-a}{\gamma}$

and $T^+ = \frac{S+a}{\gamma}$. For convenience we assume that $\frac{S-a}{\gamma}$ and $\frac{S+a}{\gamma}$ are integer numbers, which also implies that $T^+ - T^- = \frac{2a}{\gamma}$ is a strictly positive integer (note that $\frac{2a}{\gamma} \geq 1$, so that $a \geq \frac{\gamma}{2}$).

3.2 Adjustment hazard

Here, we characterize the distribution of price changes. In our simple set-up, the size of price change is γt if price changes at date t, so the size of price change is an obvious function of the duration of the first price spell. We note τ this random variable, which is the waiting time before hitting the threshold. We then use the hazard function approach to derive the distribution of τ. This approach is insightful in the present context. The probability of price change after t periods can be written as the product of the probability of observing no price change for $t-1$ periods and the conditional probability of price change after t periods. The former term is the survival function $s(t)$ and the latter one is the hazard function $h(t)$. Formally,

$$
\begin{aligned}
P(\Delta p_t &\neq 0, p_{t-1} = ... = p_0) \\
&= P(p_{t-1} = ... = p_0) \times P(\Delta p_t \neq 0 | p_{t-1} = ... = p_0) \\
&= s(t) \times h(t)
\end{aligned}
$$

As observed before, the price can only be modified between dates T^- and T^+. So, if $t \leqslant T^-$ then $s(t) = 1$ and $t > T^+$ then $s(t) = 0$. If $t \in [T^-; T^+]$, the survival function is the probability that the "high threshold" has been realized for $t - T^-$ periods, that is:

$$
\begin{aligned}
s(t) &= P(p_{t-1} = ... = p_{T^-}) \\
&= \left(\frac{1}{2}\right)^{t-T^-}
\end{aligned}
$$

The hazard function is:

$$
\begin{aligned}
h(t) &= P(\Delta p_t \neq 0 | p_{t-1} = ... = p_0) \\
&= P(p_t^* - p_0 \geq S_t) \\
&= P\left(\gamma t \geq S + \nu_t\right)
\end{aligned}
$$

Using $T^- = \frac{S-a}{\gamma}$,

$$\begin{aligned} h(t) &= P\left(\gamma t \geq \gamma T^- + a + \nu_t\right) \\ &= P\left(\nu_t \leqslant -a + \gamma(t - T^-)\right) \end{aligned}$$

Given the process for ν_t, the hazard function can only take three values in our model $\{0, \frac{1}{2}, 1\}$. For $t < T^-$, $h(t) = 0$. For $T^- \leq t < T^+$, $h(t) = \frac{1}{2}$. In that case indeed, at each period, given that the price has not be changed for $(t-1)$ periods, the probability of price change is $\frac{1}{2}$ (the probability of hitting the lower the band). Last, if $t = T^+$, $h(t) = 1$ since $-a+\gamma(t-T^-) = a$, so that $P\left(\nu_t \leqslant -a + \gamma(t - T^-)\right) = 1$. In this case, the current price deviation hits the upper limit of the stochastic band, and the probability of price change is equal to one.

To summarize, the hazard function is the following stepwise function:

$$\begin{aligned} h(t) &= 0 && \text{if } t < (S-a)/\gamma \\ h(t) &= \frac{1}{2} && \text{if } (S-a)/\gamma \leq t < (S+a)/\gamma \\ h(t) &= 1 && \text{if } t = (S+a)/\gamma \end{aligned}$$

We can obtain the distribution of the waiting time τ as the product of $h(t)$ and $s(t)$.

$$\begin{aligned} P(\tau &= t) = 0 && \text{if} \quad t < (S-a)/\gamma \\ P(\tau &= t) = \left(\frac{1}{2}\right)^{t - \frac{S-a}{\gamma} + 1} && \text{if} \quad (S-a)/\gamma \leq t < (S+a)/\gamma \\ P(\tau &= t) = \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} && \text{if} \quad t = (S+a)/\gamma \end{aligned}$$

It will be useful to consider the distribution of waiting time once date T^- is elapsed, that is to consider the random variable $\tilde{\tau} = \tau - T^- + 1$. The distribution of $\tilde{\tau}$ is as follows:

$$\begin{aligned} P(\tilde{\tau} &= k) = 0 && \text{if} \quad k < 1 \\ P(\tilde{\tau} &= k) = \left(\frac{1}{2}\right)^{k} && \text{if} \quad 1 \leqslant k < \frac{2a}{\gamma} + 1 \\ P(\tilde{\tau} &= k) = \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} && \text{if} \quad k = \frac{2a}{\gamma} + 1 \end{aligned}$$

Note this distribution is a geometric distribution with parameter $\frac{1}{2}$, that is truncated at value $\frac{2a}{\gamma} + 1$.

Before proceeding, we can relate the above result to the notion of adjustment hazard that has been introduced by Caballero and Engel (1993 and 1999) to analyze generalized (S, s) models. In such models, the probability of a price change depends on the gap z_t between the optimal price at date t and the current price at that date, i.e. $z_t = p_t^* - p_0$. The probability of a price change expressed as a function of gap variable is called the adjustment hazard function. In our simple framework here $z_t = \gamma t$, so the adjustment hazard is simply $\Lambda(z_t) = h(z_t/\gamma)$. From above, we see that the adjustment hazard is non-decreasing here. This matches the "increasing hazard" property: Caballero and Engel (1993 and 1999) have shown that the adjustment hazard increases with z_t in generalized (S, s) models.

We can also describe how the adjustment hazard varies with the average threshold S and the band variability a. The hazard is a non-increasing function of S. For large values of S, the range of price gaps z_t's for which the hazard function is null is wider, whereas the range of values of z_t for which the hazard function is 0.5 or 1 is moved to the right. Variations in a, the standard deviation of S_t, have more complicated effects on the hazard. For larger values of a, the range of z_t's for which the hazard function is equal to 0.5 broadens whereas the range of z_t for which the hazard function is 0 is narrower. At the same time, the set of z_t's for which the hazard function is equal to 1 is moved to the right. An extreme case appears when a is very large, implying large variations in S_t. In that case, the adjustment hazard is constant (equal to $\frac{1}{2}$) and the probability of price change does not depend on z_t, as in the Calvo (1983) model. Another extreme case is $a = 0$, $S_t = S$: $\Lambda(z_t)$ is then equal to 1 when $z_t = S$ and 0 otherwise. This case corresponds to the standard (S, s) model with time-invariant bands. Overall, we observe that increasing a flattens the adjustment hazard function.

3.3 Moments of price changes

In this section, we compute the different moments of the size of price changes, conditional on a price change being observed.

3.3.1 Average price change

We first compute the mean of price change, when a price change is implemented. The first moment of price changes is a function of parameters, which we denote $m_1(S, a, \gamma)$. Given the distribution of the price duration τ obtained in the previous section, we have:

$$\begin{aligned}
m_1(S,a,\gamma) &= E(\Delta p_\tau | p_\tau \neq p_{\tau-1}, p_{\tau-1} = ... = p_0) \\
&= E(\gamma\tau) \\
&= E\left(\gamma\left(\widetilde{\tau} + T^- - 1\right)\right) \\
&= E\left(\gamma\left(\frac{S-a}{\gamma} - 1\right) + \gamma\widetilde{\tau}\right) \\
&= S - a - \gamma + \gamma\left[\sum_{k=1}^{\frac{2a}{\gamma}} k\left(\frac{1}{2}\right)^k + \left(\frac{2a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right]
\end{aligned}$$

After some algebra we obtain an analytical expression appearing in Proposition 1.

Proposition 1

$$m_1(S,a,\gamma) = S - a + \gamma\left(1 - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)$$

Dem: see Appendix

From proposition 1, we derive three properties:

Properties

(1.1)

$$\frac{\partial m_1(S,a,\gamma)}{\partial S} = 1 > 0$$

(1.2) Under assumption that $a > \frac{\gamma}{2}$,

$$m_1(S,a,\gamma) < S$$

(1.3)

$$\frac{\partial m_1(S,a,\gamma)}{\partial a} < 0$$

Property (1.1) is straightforward. Properties (1.2) and (1.3) are demonstrated in the Appendix. Assumption $a > \frac{\gamma}{2}$ derives from $a > 0$.

Properties (1.1) and (1.2) relate the average size of price changes to S. As expected, the average price change increases with the value of the threshold. Also, Property 1.2 indicates that the average size of price increases is lower when there is time variation in the band ($a > 0$) than when the band is deterministic ($a = 0$).

Property (1.3) characterizes the relation between average size of price changes and the standard deviation of the band. Recall that a is an index of the variability of the threshold since the standard deviation of S_t is equal to $\sigma_S = a$. The average size of price change decreases with the standard deviation of the band. This result is essential in our context. In particular, allowing for a large variance of the band, the model can generate a very small (here arbitrarily small) average size of observed price change.

3.3.2 Variance of price changes

We now compute the variance of the size of price changes, conditional on a price change is implemented:

$$\begin{aligned} m_2(S,a,\gamma) &= V(\Delta p_\tau | p_\tau \neq p_{\tau-1}, p_{\tau-1} = ... = p_0) \\ &= V(\gamma\tau) \\ &= \gamma^2 V\left(\left(\frac{S-a}{\gamma}-1\right)+\widetilde{\tau}\right) = \gamma^2 V(\widetilde{\tau}) \\ &= \gamma^2 \left[\sum_{k=1}^{\frac{2a}{\gamma}} k^2 \left(\frac{1}{2}\right)^k + \left(\frac{2a}{\gamma}+1\right)^2 \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2\right] \end{aligned}$$

After some algebra we obtain an analytical expression appearing in Proposition 2.

Proposition 2

$$m_2(S,a,\gamma) = \gamma^2 \left[2 - \left(\frac{4a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(\frac{1}{2}\right)^{\frac{4a}{\gamma}}\right]$$

Dem: see Appendix

From proposition 2, we derive two properties:

Properties

(2.1)

$$\frac{\partial m_2(S,a,\gamma)}{\partial S} = 0$$

(2.2)

$$\frac{\partial m_2(S, a, \gamma)}{\partial a} > 0$$

Property (2.1) is straightforward. Property (2.2) is demonstrated in the Appendix.

Property (2.1) relates the variance of price changes to S. We find that the variance of price changes is invariant to the value of the threshold S.

Property (2.2) characterizes the relation between the variance of price changes and the standard deviation of the band. We find that the variance of price changes increases with the standard deviation of the band.

To illustrate how these properties translate into actual data consider the case of a model with parameters (S, a) which generates data with mean and variance m_1 and m_2. Now, assume we observe other data characterized by the same mean $m_1^* = m_1$ but a higher variance $m_2^* > m_2$, that are postulated to be generated by a similar model and a parameter set (S^*, a^*). Given that m_2 does not depend on S, Property (2.2) indicates that the variance of the threshold is necessarily larger in the second case: $a^* > a$. Now, given Properties (1.1) and (1.3) observing the same data mean $m_1^* = m_1$ with a large band variability $a^* > a$ is only possible if $S^* > S$. A larger value of S^* balances the fact that m_1 decreases with a. This is one main message of the present paper: to fit data with a substantial variance in adjustment size (and a given mean) both a large bandwidth and a large variance of the threshold are necessary.

4 Properties of a time-varying (S, s) band models: simulation evidence

In this section, we use simulations to illustrate how the results presented above extend to other specifications of the threshold S_t that, to our knowledge, to do not lead to analytically tractable problems.

4.1 Simulation design

We focus on the case where the threshold follows a Gaussian distribution. These assumptions bring us close to specifications used in the empirical studies (see for example, Fisher and

Konieczny (2006) or Dhyne *et al.* (2007)). We build a simulation exercise where the frictionless optimal price is defined as: $p_t^* = \gamma t$ where $\gamma > 0$ and $p_0^* = 0$ and the price threshold is defined as: $S_t = S + \nu_t$ where $\nu_t \sim N(0, \sigma_\nu^2)$.[3] We define τ as the elapsed duration since last price change. The price is changed according the following rule. If $p_t^* - p_{t-\tau} \geq S_t$ (with $\tau > 1$), the observed nominal price is changed and the new price is set to $p_t = p_t^* = p_{t-\tau} + \gamma\tau$. If $p_t^* - p_{t-\tau} < S_t$, then the price stays unchanged, i.e. $p_t = p_{t-\tau}$. At each date of price change $\{t|p_t \neq p_{t-1}\}$, we compute the size of price change. We then compute the average size of non-zero price changes m_1 and the variance of price changes m_2 by simulating the process for a very large sample. This exercise is repeated for different values of S and σ_ν. Results presented in this section have been obtained with samples of size $T = 3,000,000$, and parameter values $p_0^* = 0$, $\gamma = 0.25$, $S \in [0.5; 25.0]$ and $\sigma_S \in [0.5; 10.0]$. This range of parameters produces a wide range of values for the mean and variance of price changes that encompass values reported in the existing empirical literature on micro price adjustments.

4.2 Adjustment hazard function

In this framework, the adjustment hazard function can be written as:

$$\begin{aligned} h(z_t) &= P(\Delta p_t \neq 0 | p_{t-1} = ... = p_{t-\tau}) \\ &= P(p_t^* - p_{t-\tau} > S_t) \\ &= P\left(z_t \geq S + \nu_t\right) \\ &= P\left(\gamma\tau - S \geq \nu_t\right) \\ &= \Phi\left(\frac{\gamma\tau - S}{\sigma_S}\right) \end{aligned}$$

where Φ is the cumulative distribution function of a standard normal distribution.

Then, three results appear:

(1) $\frac{\partial h(z_t)}{\partial S} < 0$. The hazard decreases with the average band, which is consistent with the findings of section 2.2.

(2) if $z_t - S < 0$ then $\frac{\partial h}{\partial \sigma_S} > 0$ and if $z_t - S > 0$ then $\frac{\partial h}{\partial \sigma_S} < 0$. Like in the discrete case, an increase in σ_S flattens the hazard adjustment function.

[3] Findings are robust to considering more general processes for p_t^*, as detailed below and in Appendix 2.

(3) Two extreme cases can be exhibited. If both σ_S and S are large (say, $\sigma_S = \kappa S$ for a positive κ and S is arbitrarily large), we obtain that $h(z_t)$ is close to a constant. The probability of price change does not depend on z_t any more and the hazard adjustment function is flat. This matches the hazard function generated by a Calvo model. The second extreme case is $\sigma_S = 0$. This is the deterministic band case: the hazard is equal to 1 when $z_t = S$ and 0 otherwise.

To illustrate these properties, we plot on Figure 3 the adjustment hazard functions obtained for different values of S and σ_S. We first notice that adjustment hazards are increasing functions of z_t (on x-axis). Thus, the probability of price change increases with the disequilibrium between the observed nominal price and the optimal price. The hazard functions are thus in accordance with hazard functions generated by structural models with stochastic menu-costs: see Caballero and Engel (1993 and 1999), in the case of price-setting, Willis (2000) and section 2 of the present paper.

Secondly, adjustment hazard functions are decreasing with S (compare the case $S = 10$ in bold line with the case $S = 16$ in thin line). If S is large, as in the case $S = 16$ the range of z_t's for which the hazard is zero is wide, and very few price changes are then observed.

Third, the adjustment hazard function flattens with σ_S. In the case $S = 10$, we can distinguish two regions: for z_t between 0 and 10, increasing σ_S from $\sigma_S = 2$ (solid line) to $\sigma_S = 4$ (dashed line) increases the hazard whereas for z_t higher than 10, increasing σ_S leads to a decrease of the hazard.

Finally, we plot an extreme case (in dotted line) where $S = 40$ and $\sigma_S = 80$. This case illustrate our observation (3) above. The hazard function is close to flat and its value does not depend on z_t any more as in a Calvo model. Note that in that case the hazard function is non-zero even for very small values of z_t, i.e small price changes are likely to be observed.

Overall, these results are quite consistent with those obtained with a more structural model in which menu cost are randomly distributed (Caballero and Engel (1999) or our model in section 2). In particular the simple DGP used here is able to generate various patterns of the adjustment hazard, that all fulfill the increasing adjustment hazard property and are consistent with a structural random menu cost model.

4.3 Moments of price changes

In Figure 4, we report how the average size of price changes m_1 depends on the model parameters S (x-axis) and σ_S (y axis). We can first observe that the average size of price changes is increasing with S, as in Property (1.1) obtained above with a Bernoulli process for S_t. In addition, Figure 4 also illustrates Property (1.3): for a given bandsize parameter, increasing the variance of the band σ_ν decreases the average size of observed price changes. This result appears more clearly on Figure 5 which is a slice in Figure 4: we set $S = 5$ and represents the relationship between m_1 (on y-axis) and σ_ν (on x-axis). The average size of price changes is a decreasing function of the variance of the stochastic band: when the band varies a lot, price adjustments tend to be smaller on average. Lastly, Figures 4 and 5 indicate that the average price change is always smaller than the mean size of the band S, reflecting Property (1.2).

Figure 6 plots the variance of price changes as a function of model parameters S (x-axis) and σ_S (y-axis). Results here are partly different from those obtained in the case of a discrete process for S_t. In the region containing large values of S and the low values of σ_S, results are consistent with Properties (2.1) and (2.2): the variance of prices changes is insensitive to the variation of S whereas increase in σ_S leads to larger values of m_2. However, in the region containing small values of S and high values of σ_S, we find that increasing σ_S decreases m_2 and also that m_2 is an increasing function of S. The intuition is that there is a maximum to the variance of price changes that can be produced with the model. Indeed, when the variance of the band is large, price changes will tend to be smaller, due to the mechanism of Property (1.3). Since all price changes are clustered in the zone of small price changes, the variance of price changes can not be as large as possible. In that zone, increasing S relaxes the constraint on the variance of price changes.

Figure 7 illustrates Properties (1.1), (1.3), (2.1) and (2.2), by representing the contour lines associated with Figures 4 and 6 on the same graph. S and σ_S are represented on the x-axis and y-axis respectively. Contour lines for m_2 are the "L-shaped" curves, with values closest to the North-East corresponding to largest values of m_2. Contour lines for m_1 are the nearly straight lines. Two contour lines are drawn $m_1 = 6$ and $m_1 = 10$. The two regions described above for

m_2 are visible from the graph. In the upper-left part of the graph, increasing σ_S leads to lower values of m_2 and increasing S leads to increase m_2. This region is characterized by lower values of m_1. In the lower-right region of the graph (say below contour line $m_1 = 6$, a zone where price changes are larger on average) m_2 does not respond to changes in S whereas it increases with σ_S. Overall, we observe in both regions that to attain a larger variance of price changes (say from $m_2 = 1$ to $m_2 = 2$) while maintaining the same average price change (for example $m_1 = 6$), the model requires both larger values of σ_S and S.

To further illustrate this last result, we perform a moment matching exercise. We set $m_1 = 6.$ and we consider values of m_2 ranging 1 and 3. We then identify the underlying parameters σ_S and S through a minimum distance procedure. Namely for each candidate value (S, σ_S), simulated moments $\widetilde{m}_1(S, \sigma_S)$ and $\widetilde{m}_2(S, \sigma_S)$ are computed using the same simulation exercise as described above (with a trajectory of size $T = 3 \times 10^6$) and a numerical optimization routine is used to set the distance $(m_1 - \widetilde{m}_1(S, \sigma_S))^2 + (m_1 - \widetilde{m}_1(S, \sigma_S))^2$ to zero. Figure 8 plots values of S obtained by this simulated method of moments, as a function of the target moment m_2 (with $m_1 = 6$). As we noticed before, given m_1, matching a larger values of m_2, requires larger values of both σ_S and S.

Results in this section were obviously obtained from specific processes for p_t^* and S_t. However, as mentioned above, results are robust to considering alternative processes. In Appendix 2, we provide some results obtained for a different process for p_t^*, we have now $p_t^* = \gamma t + \varepsilon_t$ where $\gamma > 0$ and $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ (we perform different exercises with $\sigma_\varepsilon = 1$, $\sigma_\varepsilon = 2$ and $\sigma_\varepsilon = 4$), the rest of the simulation exercise remains the same. We can observe that all our baseline results obtained with $\sigma_\varepsilon = 0$ are not qualitatively different from the ones obtained with $\sigma_\varepsilon \neq 0$. It is nevertheless obvious that for very large values of σ_ε, the influence of σ_S on the results becomes weaker. We conjecture that Properties (1.1) to (2.2) are valid for a very wide range of processes, though further analytical investigation of these issues is left for future research.

5 Interpreting actual estimates

In this last section, we use actual estimation results from Dhyne, Fuss, Pesaran and Sevestre (2007) to illustrate the properties investigated above. Dhyne *et al.* (2007) estimate an (S, s) model with stochastic bands using individual price quotes of more than two hundred products (sold in Belgium and France). Their model is very close to the one presented in Section 3. For each product, the frictionless optimal price in their framework is defined as $p_t^* = f_t + \varepsilon_t$ where f_t is a common factor representing a common component to all outlets and ε_t is an idiosyncratic shock defined as $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. The price change rule is the following: a price is modified as soon as $|p_t^* - p_\tau| > S_t$ where $S_t = S + \nu_t$ with τ the date of the last price change and $\nu_t \sim N(0, \sigma_\nu^2)$ (see Dhyne *et al.* (2007) for details).

Dhyne *et al.* (2007) report results of estimated values of $\sigma_{\nu i}$ and S_i as well as the actual average price change $m_{1i} = E(\Delta p_t | p_t \neq p_{t-1}, p_t = ... = p_0)$ for a cross section of products indiced by i. We restrict our sample to products for which the proportion of price increases is greater than 70% (56 products), to get closer to the framework we presented above with only price increases. We use results reported in their paper (Tables A and B of their appendix) to plot Figure 9 which is the superposition of two (cross-sectional) scatter plots: the average of observed price change m_{1i} and the size of the band S_i, both as a function of the standard deviation of the band $\sigma_{\nu i}$.

First, we observe that for all products m_{1i} are inferior to S_i. The difference between the two values can be very large, the median of m_{1i}s is close to 5% whereas the median of estimated S_i's is closer to 40%. This result illustrates Property 1.2. Figure 9 also indicates that for higher standard deviations of the band $\sigma_{\nu i}$, the gap between the size of band S_i and the average price change m_{1i} increases. For large values of $\sigma_{\nu i}$, the estimated size of the band is clearly not informative about the observed average price change.

Based on our previous results, an interpretation for the gap observed in the right most part of Figure 9 emerges. Observations on the rightmost part of the figure are characterized by a large value of $\sigma_{\nu i}$, and therefore presumably associated with sectors with a high variance of price changes. Variability in the size of price adjustment is a well documented fact in micro price data.

From our simulation exercise, we know that to fit a large variability in price changes m_{2i}, while the average size of price changes m_{1i} is the roughly the same for all products, the procedure needs to assume a large bandsize parameter S_i. Our interpretation is that a large level and variance of the band are here needed in order to match the variance of price changes.

An alternative insight may be obtained from the adjustment hazard function. On Figure 3, we have included the hazard function associated to $S = 40$ and $\sigma_S = 20$, which correspond to the median values of the scatter plot of Figure 9. For these values, the hazard is nearly flat but non-zero near $t = 1$, so that small price changes are allowed. Consistent with result in section 3 and 4, large estimates of σ_S and S here reflect the prevalence of small price change, and that fact that for some items the hazard function has a Calvo (flat) pattern.

Large estimates of S suggest that the average menu cost is very substantial. Using the structural model with stochastic adjustment costs in section 2, provides additional insights. Figure 10 plots the distributions of menu-costs when prices change and when prices are kept unchanged. One could observe that these two distributions are different. The average menu-cost at price changes is equal to 4.89%, whereas the average menu cost for dates when no price change is observed is 6.00%. Firms adjust their price when they face lower adjustment costs. As a result, the distribution of adjustment costs is not informative about patterns of adjustment costs actually observed when prices are changed (see Willis (2000) for very similar results on magazine prices).

6 Conclusion

We have illustrated some properties of (S, s) models with time-varying random thresholds, an empirical specification increasingly used to model data featuring infrequent adjustment. First, the adjustment hazard is shown to decrease with the size of the threshold and to flatten with the variance of the band. Second, this model is able to produce a large variety of hazard functions. Two polar cases are the constant hazard model (Calvo, 1983) (for large values of both S and σ_S) and the traditional fixed band (S, s) model (for $\sigma_S = 0$). Third, the average size of price changes is an increasing function of the mean band size, and is always smaller than this band.

The average size of price changes decreases with the variance of the band. Finally, the variance of size of adjustment generated by the model is not sensitive to the mean of the band, but is an increasing function of the variance of the band.

These results have important implications for the interpretation of stochastic (S, s) model estimates. An abundant empirical literature dealing with investment, durable consumption, hiring or price-setting decisions report a high variability of the size of adjustments at the microeconomic level. Estimating (S, s) models with time-varying random thresholds is an appealing solution in this context. As illustrated by our results this flexible specification indeed allows to match data featuring infrequent microeconomic adjustments of variable size. Our results however show that to match a large variance of adjustments for a given mean of these adjustments, these models need to produce large estimates for bandsize parameters. Contrary to the core (S, s) model with fixed bands, the estimated size of the band is then not anymore informative about the size of actual adjustments. Large estimated average threshold (and menu costs) do not imply the econometric rejection of the model. Those cases may however raise the question of whether substantial fluctuations in menu costs are an economically plausible mechanism.

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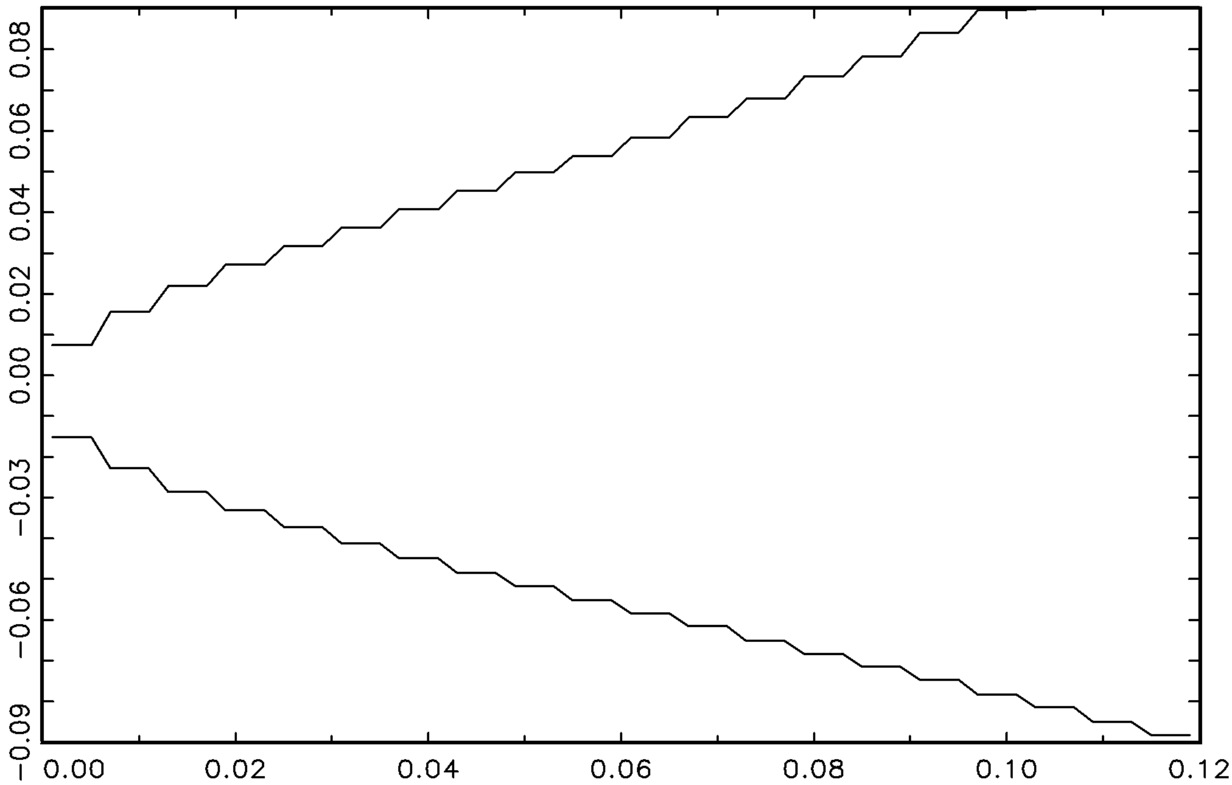
Figure 1: Policy function of the structural random menu cost model



Note: x-axis: values of the menu-cost; y-axis: values of $z_t = p_{t-\tau} - p_t^*$, where τ is the duration elapsed since the last price change.

Figure 2: Trajectory of the price deviation

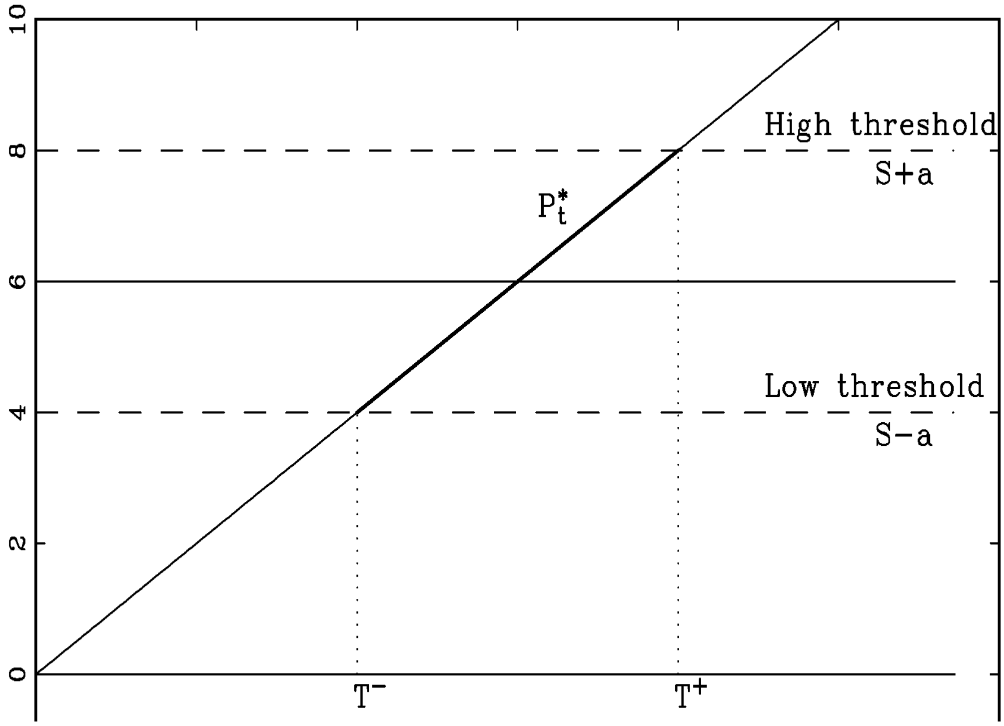
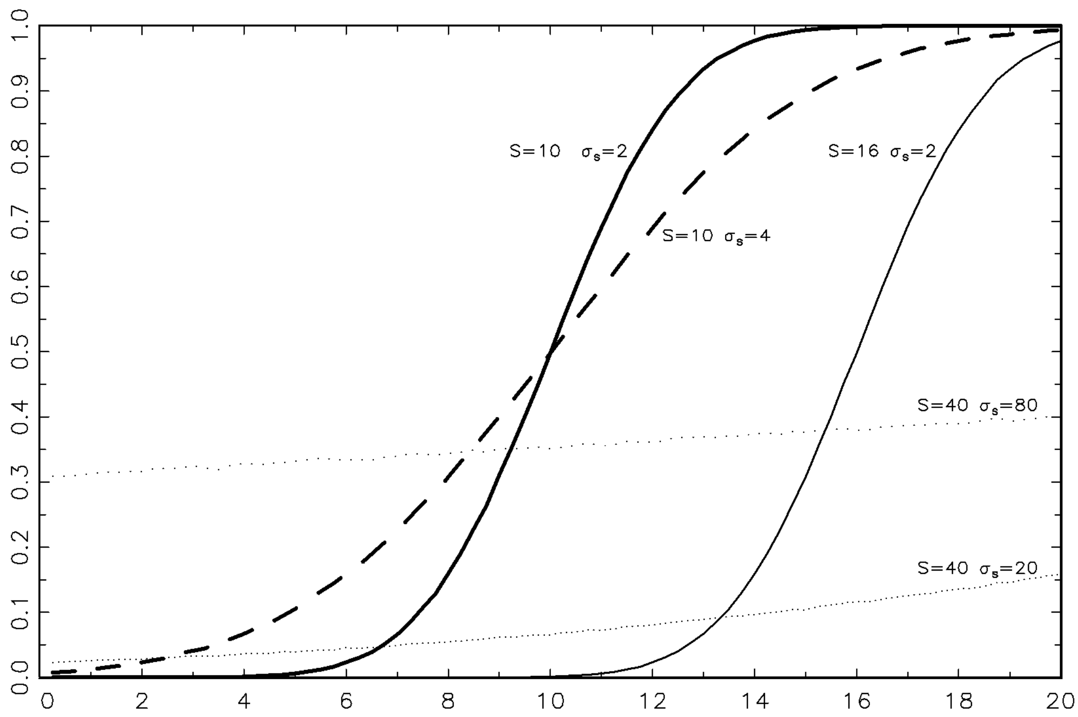
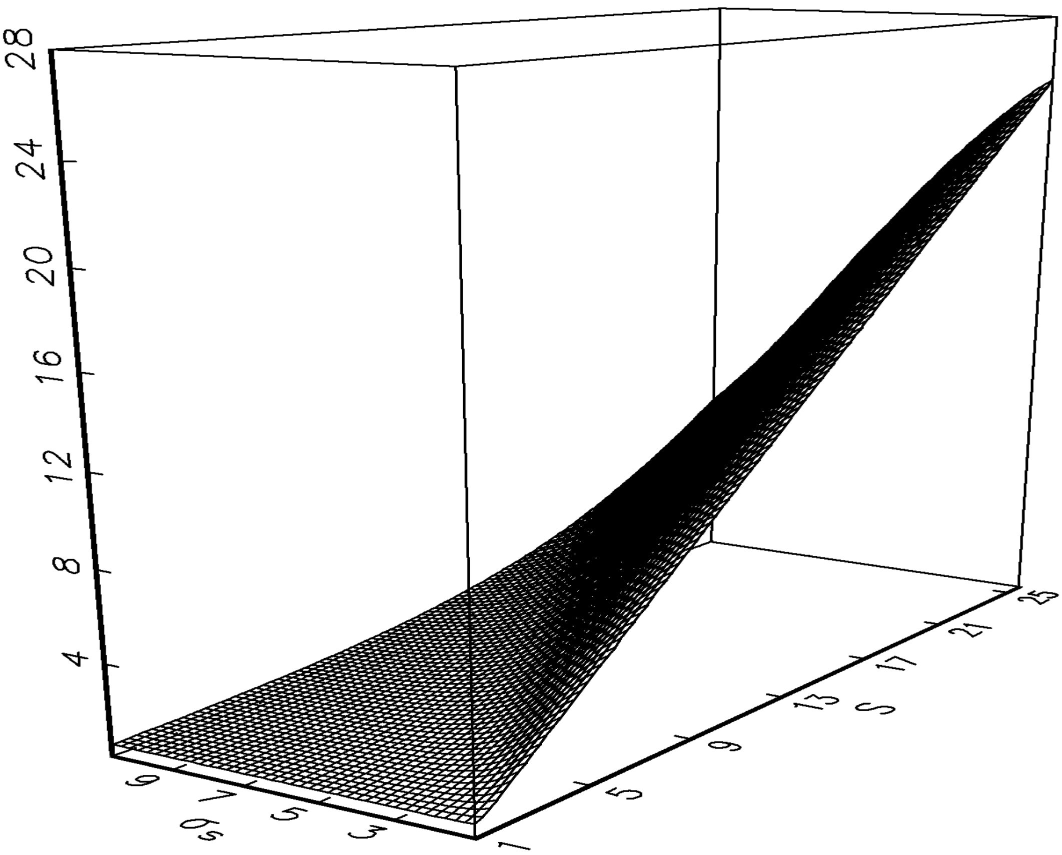


Figure 3: Adjustment hazard as a function of model parameters (S, σ_S)



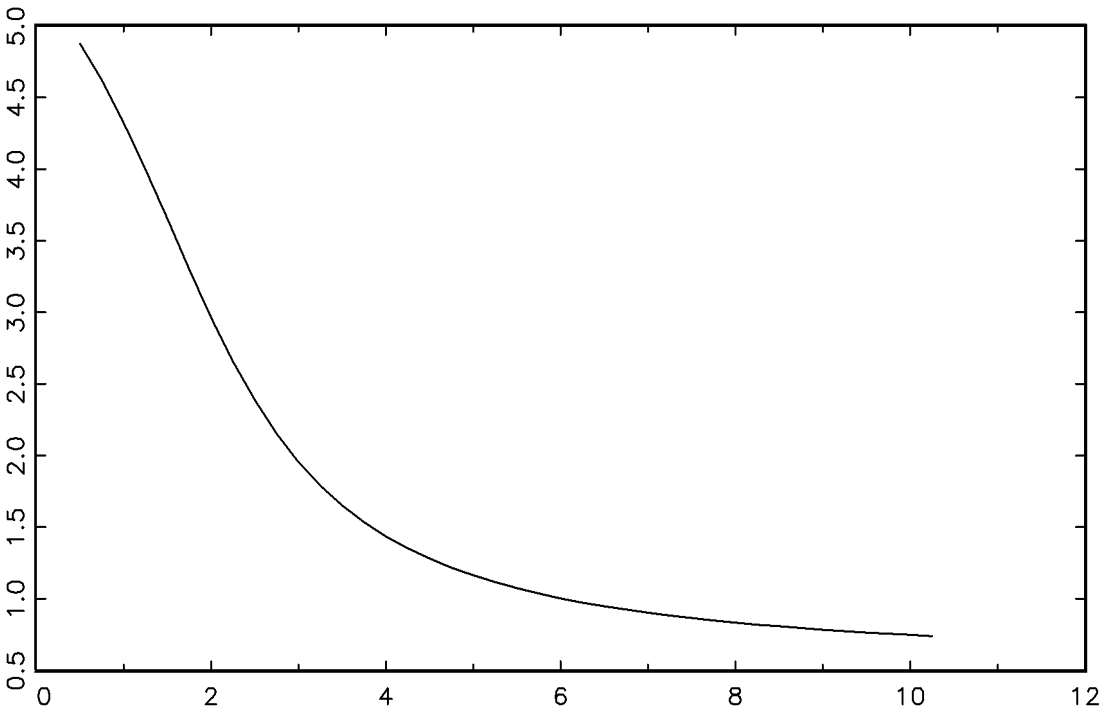
Note: x-axis: values of $z_t = p_{t-\tau} - p_t^*$, where τ is the duration elapsed since the last price change. y-axis: the probabitly of price change. Each line is generated for different values of S and σ_S.

Figure 4: The average size of price change as a function of model parameters (S, σ_S)



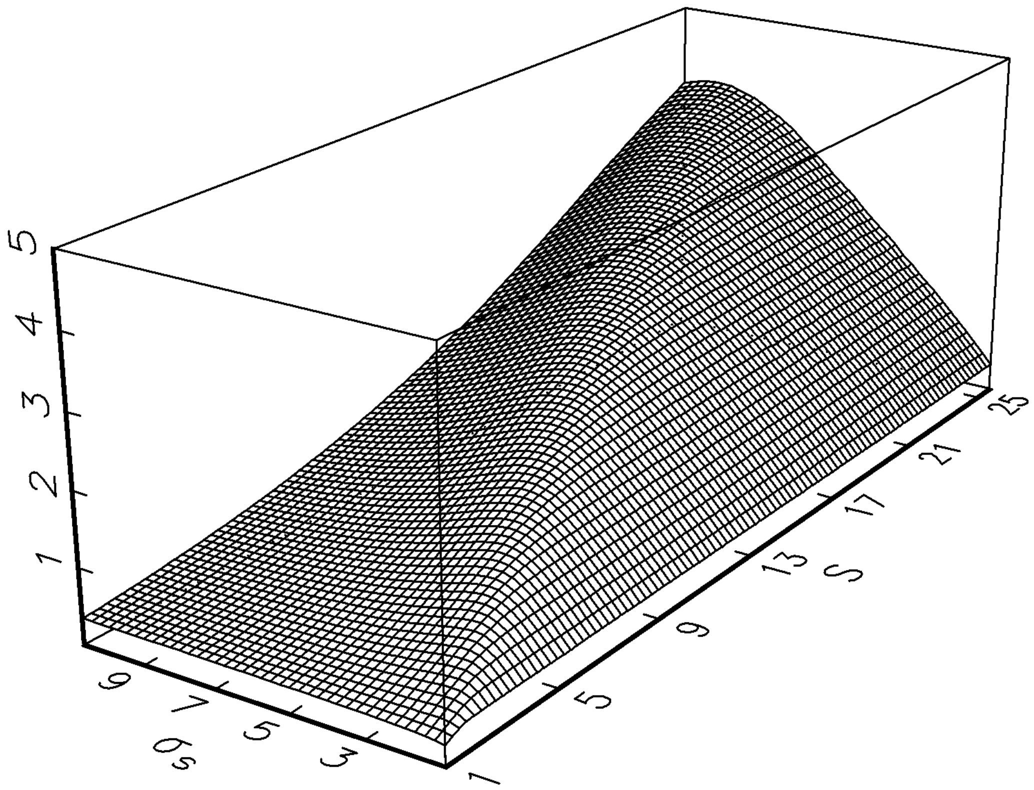
Note: x-axis: σ_S the standard deviation of S_t y-axis: S the average of S_t and z-axis: m_1 the average size of price changes.

Figure 5: The average size of price changes m_1 as a function of the variance of the band σ_S ($S = 5$)



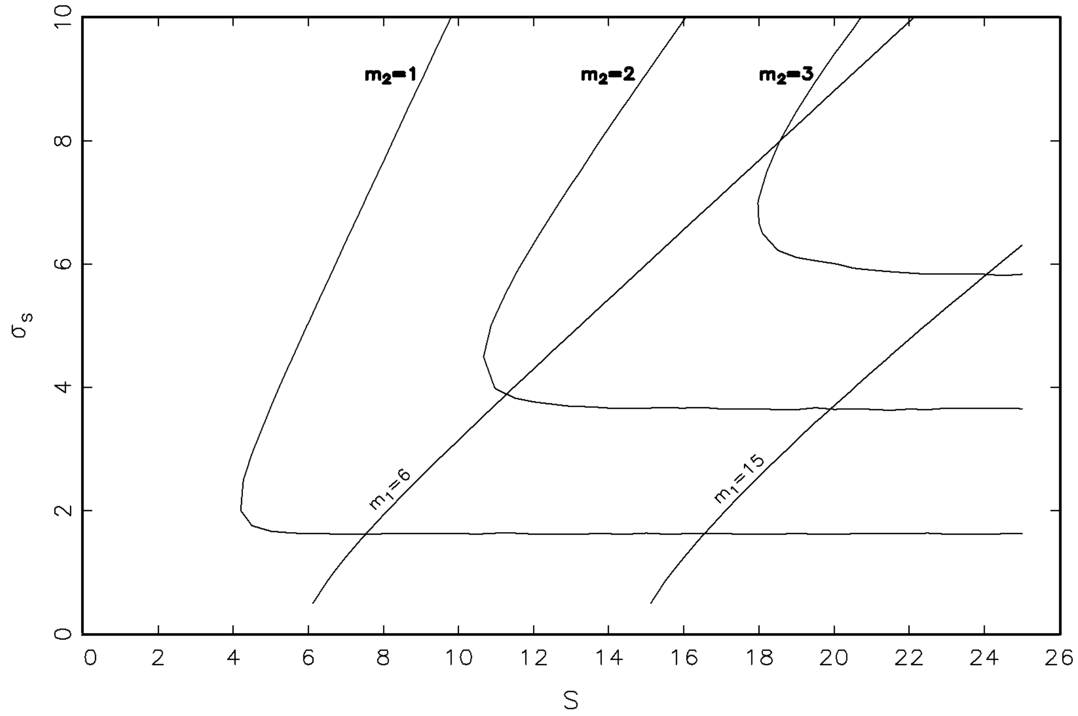
Note: x-axis: σ_S the standard deviation of S_t y-axis: m_1 the average size of price changes.

Figure 6: Standard deviation of price changes as a function of model parameters (S, σ_S)



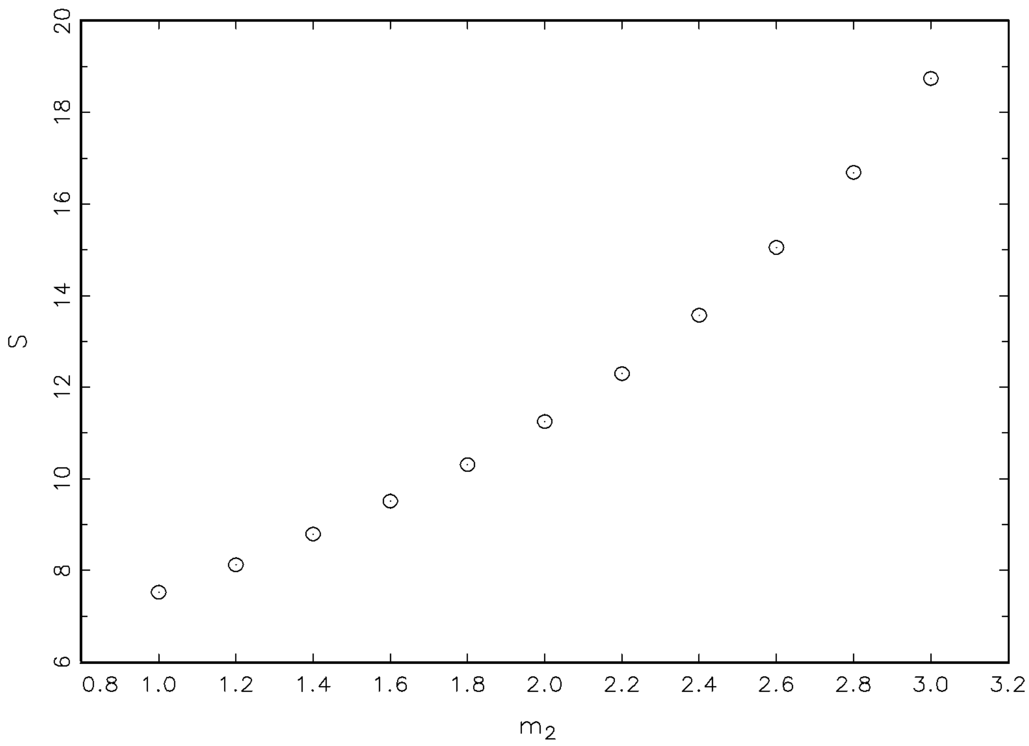
Note: x-axis: σ_S the standard deviation of S_t y-axis: S the average of S_t and z-axis: m_2 the standard deviation of the size of price changes.

Figure 7: Contour plots of mean and variance of price changes in the plane (S, σ_S)



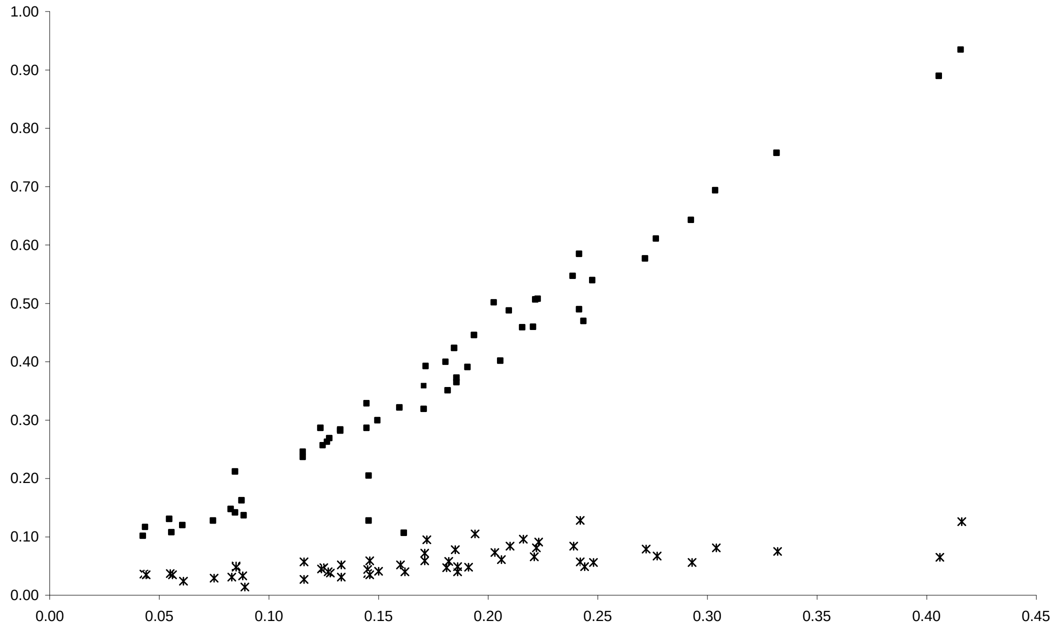
Note: x-axis: S the average of S_t and y-axis: σ_S the standard deviation of S_t. Each line is a contour plot for different values of m_1 and m_2.

Figure 8: Values of S matching moments ($m_1 = 6$ and $m_2 \in [1;3]$) - Minimum distance estimates



Note: x-axis: m_2 the standard deviation of the size of price changes and y-axis: S the average of S_t.

Figure 9: Average price changes and S as a fuction of the variance of the band (source: Dhyne *et al.* (2007))



Note: for each sector, the average price change (crosses - y-axis) and the average size of the band (black squares - y-axis) are plotted against the residual variance of the band (x-axis).

Figure 10: Distribution of menu-costs when price change and when prices are kept unchanged

Note: white bars: distribution of menu costs at price changes and black bars: menu costs when prices remain unchanged.

8 Appendix 1

Proposition 1

$$\begin{aligned}
m_1(S,a,\gamma) &= S-a-\gamma+\gamma\left[\sum_{k=1}^{\frac{2a}{\gamma}} k\left(\frac{1}{2}\right)^{k}+\left(\frac{2a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right] \\
&= S-a-\gamma+\gamma\left[\frac{1}{2}\sum_{k=1}^{\frac{2a}{\gamma}} k\left(\frac{1}{2}\right)^{k-1}+\left(\frac{2a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right] \\
&= S-a-\gamma+\gamma\left[\frac{4}{2}\left(1-\left(\frac{a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)+\left(\frac{2a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right] \\
&= S-a-\gamma+\gamma\left[2-2\left(\frac{a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}+2\left(\frac{a}{\gamma}+\frac{1}{2}\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right] \\
&= S-a-\gamma+\gamma\left[2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right] \\
&= S-a+\gamma\left[1-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right]
\end{aligned}$$

In the above we have used the following property.

We define $G(x)=\sum_{j=0^-}^{J} x^j = \frac{1-x^{J+1}}{1-x}$

Then: $G'(x)=\sum_{j=0}^{J} jx^{j-1} = \frac{(x-1)(J+1)x^J+1-x^{J+1}}{(1-x)^2}$

We have $x=\frac{1}{2}$ then

$$\begin{aligned}
G'\left(\frac{1}{2}\right) &= \frac{-(J+1)\left(\frac{1}{2}\right)^{J+1}+1-\left(\frac{1}{2}\right)^{J+1}}{\left(1-\frac{1}{2}\right)^2} \\
&= 4\left[\left(\frac{1}{2}\right)^{J+1}(-1-J-1)+1\right] \\
&= 4-\left(\frac{1}{2}\right)^{J-1}(J+2)
\end{aligned}$$

If $J=\frac{2a}{\gamma}$

$$G'\left(\frac{1}{2}\right) = 4 \times \left(1 - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} \left(\frac{a}{\gamma} + 1\right)\right)$$

Property 1.2

Let us define $f(a) = \gamma\left(1 - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right) - a$

Then,

$$\frac{\partial f}{\partial a} = 2\ln 2 \times \exp\left[-\frac{2\ln 2}{\gamma}a\right] - 1$$

$$\frac{\partial^2 f}{\partial a^2} = -\frac{(2\ln 2)^2}{\gamma} \times \exp\left[-\frac{2\ln 2}{\gamma}a\right] < 0$$

We have also $\frac{\partial f}{\partial a}\left(\frac{\gamma}{2}\right) = \ln 2 - 1 < 0$ and $\lim_\infty \frac{\partial f}{\partial a}(a) = -1$.

Then, f is a decreasing function of a. We also find that $f(\frac{\gamma}{2}) = \gamma\left(1 - \frac{1}{2}\right) - \frac{\gamma}{2} = 0$. We can then conclude that $m_1(S, a, \gamma) < S$ for all values of a different from $\frac{\gamma}{2}$ and $m_1(S, a, \gamma) = S$ if $a = \frac{\gamma}{2}$.

Property 1.3

$$\begin{aligned} m_1(S, a, \gamma) &= S - a + \gamma\left(1 - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right) \\ &= S - a + \gamma\left(1 - \exp\left[-\frac{2\ln 2}{\gamma}a\right]\right) \end{aligned}$$

Then,

$$\frac{\partial m_1(S, a, \gamma)}{\partial a} = 2\ln 2 \times \exp\left[-\frac{2\ln 2}{\gamma}a\right] - 1$$

Let define $g(a) = 2\ln 2 \times \exp\left[-\frac{2\ln 2}{\gamma}a\right] - 1$

$$g'(a) = -\frac{(2\ln 2)^2}{\gamma} \times \exp\left[-\frac{2\ln 2}{\gamma}a\right] < 0$$

We have also $g\left(\frac{\gamma}{2}\right) = \ln 2 - 1 < 0$ and $\lim_\infty g(a) = -1$.

The main result is then the following:

$$\frac{\partial m_1(S, a, \gamma)}{\partial a} < 0 \quad \text{for all } a \geq \frac{\gamma}{2}$$

Proposition 2

$$m_2(S,a,\gamma) = \gamma^2\left[\sum_{k=1}^{\frac{2a}{\gamma}} k^2\left(\frac{1}{2}\right)^k + \left(\frac{2a}{\gamma}+1\right)^2\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2\right]$$

where $(E(\widetilde{\tau}))^2 = \left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2$

$$\begin{aligned}
m_2(S,a,\gamma) &= \gamma^2\left[\sum_{k=1}^{\frac{2a}{\gamma}} k^2\left(\frac{1}{2}\right)^k + \left(\frac{2a}{\gamma}+1\right)^2\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2\right] \\
&= \gamma^2\left[\frac{1}{4}\sum_{k=1}^{\frac{2a}{\gamma}} k(k-1)\left(\frac{1}{2}\right)^{k-2} + \frac{1}{2}\sum_{k=1}^{\frac{2a}{\gamma}} k\left(\frac{1}{2}\right)^{k-1} + \left(\frac{2a}{\gamma}+1\right)^2\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2\right] \\
&= \gamma^2\left[\begin{array}{c} \frac{1}{4}\left(16-\left(\left(\frac{2a}{\gamma}\right)^2+\frac{6a}{\gamma}+4\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}-2}\right) + 2\left(\frac{2a}{\gamma}\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}+1} - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\left(\frac{2a}{\gamma}+1\right)+1\right) \\ +\left(\frac{2a}{\gamma}+1\right)^2\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2 \end{array}\right] \\
&= \gamma^2\left[\begin{array}{c} 4-\left(\left(\frac{2a}{\gamma}\right)^2+\frac{6a}{\gamma}+4\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} + \frac{2a}{\gamma}\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\left(\frac{4a}{\gamma}+2\right)+2 \\ +\left(\frac{2a}{\gamma}+1\right)^2\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2 \end{array}\right] \\
&= \gamma^2\left[\begin{array}{c} 4+2+\left(-\left(\frac{2a}{\gamma}\right)^2+\left(\frac{2a}{\gamma}\right)^2-\frac{6a}{\gamma}+\frac{2a}{\gamma}-\frac{4a}{\gamma}+\frac{4a}{\gamma}-4-2+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} \\ -\left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2 \end{array}\right] \\
&= \gamma^2\left[6+\left(-\frac{4a}{\gamma}-5\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - 4 + 4\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(\frac{1}{2}\right)^{\frac{4a}{\gamma}}\right] \\
&= \gamma^2\left[2-\left(\frac{4a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(\frac{1}{2}\right)^{\frac{4a}{\gamma}}\right]
\end{aligned}$$

In the above we have used the following property.

We find $G'(x) = \sum_{j=0}^{J} jx^{j-1} = \frac{(x-1)(J+1)x^J+1-x^{J+1}}{(1-x)^2}$

Then: $G''(x) = \sum_{j=0}^{J} j(j-1)x^{j-2} = \frac{2G'(x)-J(J+1)x^{J-1}}{1-x}$

We have $x = \frac{1}{2}$ then

$$\begin{aligned}
G''\left(\frac{1}{2}\right) &= \frac{2G'\left(\frac{1}{2}\right) - J(J+1)\left(\frac{1}{2}\right)^{J-1}}{\frac{1}{2}} \\
&= 4G'\left(\frac{1}{2}\right) - 2J(J+1)\left(\frac{1}{2}\right)^{J} \\
&= 16 - 4\left(\frac{1}{2}\right)^{J-1}(J+2) - 2J(J+1)\left(\frac{1}{2}\right)^{J-1} \\
&= 16 - (J^2+3J+4)\left(\frac{1}{2}\right)^{J-2}
\end{aligned}$$

If $J = \frac{2a}{\gamma}$

$$G''\left(\frac{1}{2}\right) = 16 - \left[\left(\frac{2a}{\gamma}\right)^2 + \frac{6a}{\gamma} + 4\right]\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}-2}$$

Property 2.2

$$\begin{aligned}
m_2(S,a,\gamma) &= \gamma^2\left[2 - \left(\frac{4a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(\frac{1}{2}\right)^{\frac{4a}{\gamma}}\right] \\
&= \gamma^2\left[2 - \left(\frac{4a}{\gamma}+1\right)\exp\left(-\frac{2\ln 2}{\gamma}a\right) - \exp\left(-\frac{4\ln 2}{\gamma}a\right)\right]
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial m_2(S,a,\gamma)}{\partial a} &= \gamma^2\left[\begin{array}{c} -\frac{2\ln(2)}{\gamma}\left(\frac{4a}{\gamma}+1\right)\exp\left(-\frac{2\ln 2}{\gamma}a\right) + \frac{4}{\gamma}\exp\left(-\frac{2\ln 2}{\gamma}a\right) \\ +\frac{4\ln(2)}{\gamma}\exp\left(-\frac{4\ln 2}{\gamma}a\right) \end{array}\right] \\
&= 2\gamma\exp\left(-\frac{2\ln 2}{\gamma}a\right)\left[-\frac{4a\ln(2)}{\gamma} - \ln(2) + 2 + 2\ln(2)\exp\left(-\frac{2\ln 2}{\gamma}a\right)\right]
\end{aligned}$$

Let define $h(a) = -\frac{4a\ln(2)}{\gamma} - \ln(2) + 2 + 2\ln(2)\exp\left(-\frac{2\ln 2}{\gamma}a\right)$

$$\frac{\partial h(a)}{\partial a} = -\frac{4\ln(2)}{\gamma} - \frac{(2\ln 2)^2}{\gamma}\exp\left(-\frac{2\ln 2}{\gamma}a\right) < 0$$

$h\left(\frac{\gamma}{2}\right) = 2(1-\ln(2)) > 0$ and $\lim_{\infty} h(a) = 0$

The main result is then the following:

$$\frac{\partial m_2(S,a,\gamma)}{\partial a} > 0 \quad \text{for all } a \geq \frac{\gamma}{2}$$

9 Appendix 2

Figure A: The average size of price change as a function of model parameters (S, σ_S) for different values of σ_ε

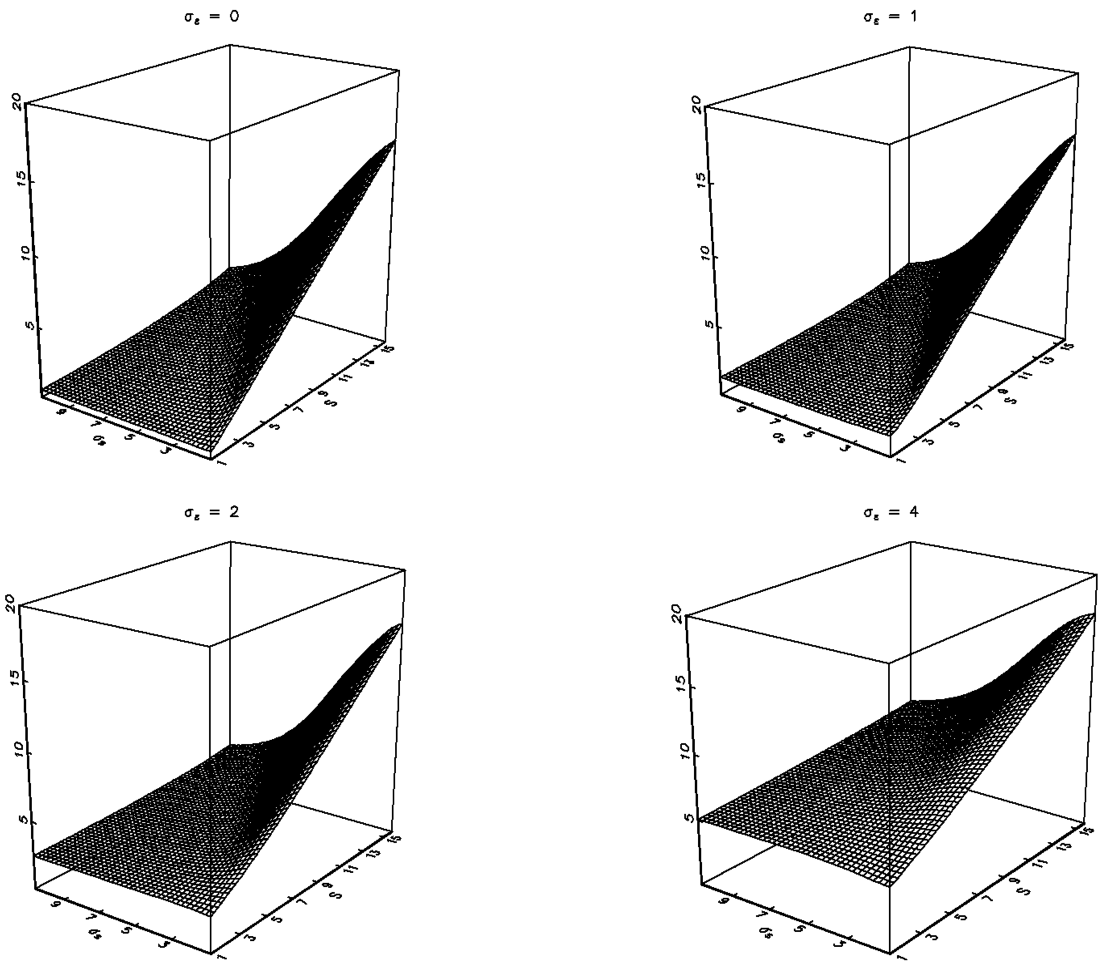


Figure B: The average size of price changes m_1 as a function of the variance of the band σ_S $(S = 5)$ for different values of σ_ε

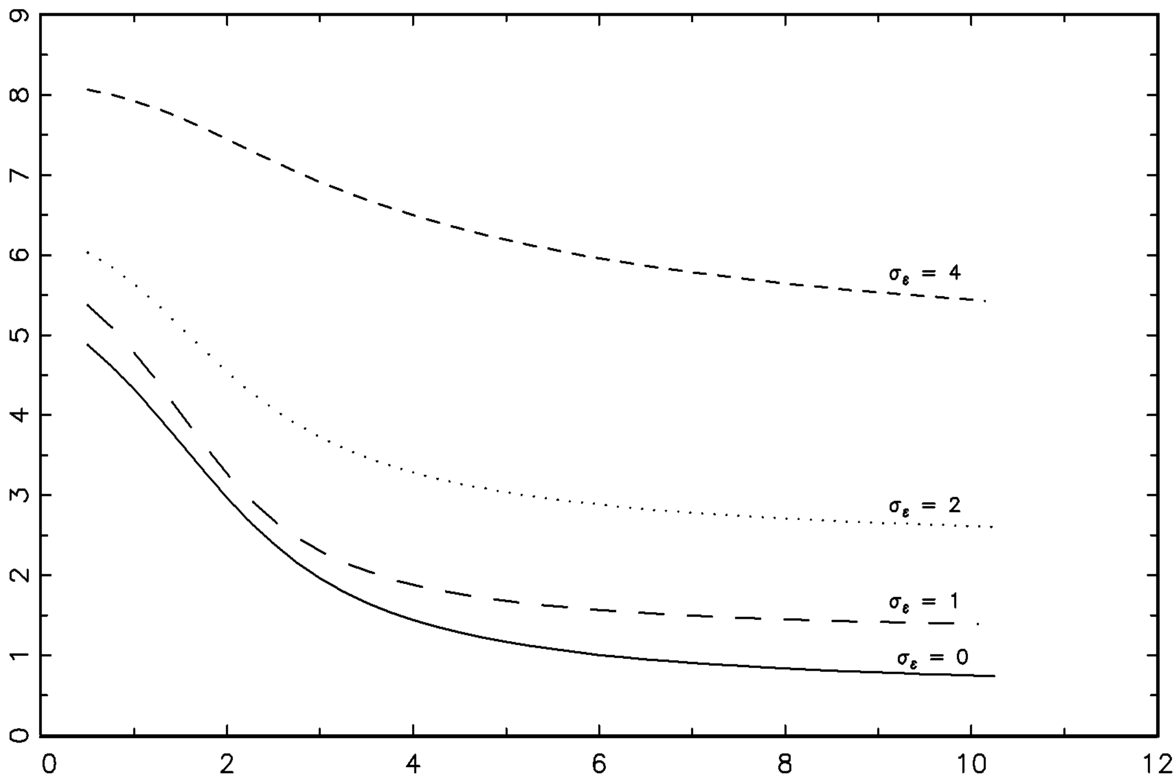


Figure C: Standard deviation of price changes as a function of model parameters (S, σ_S) for different values of σ_ε

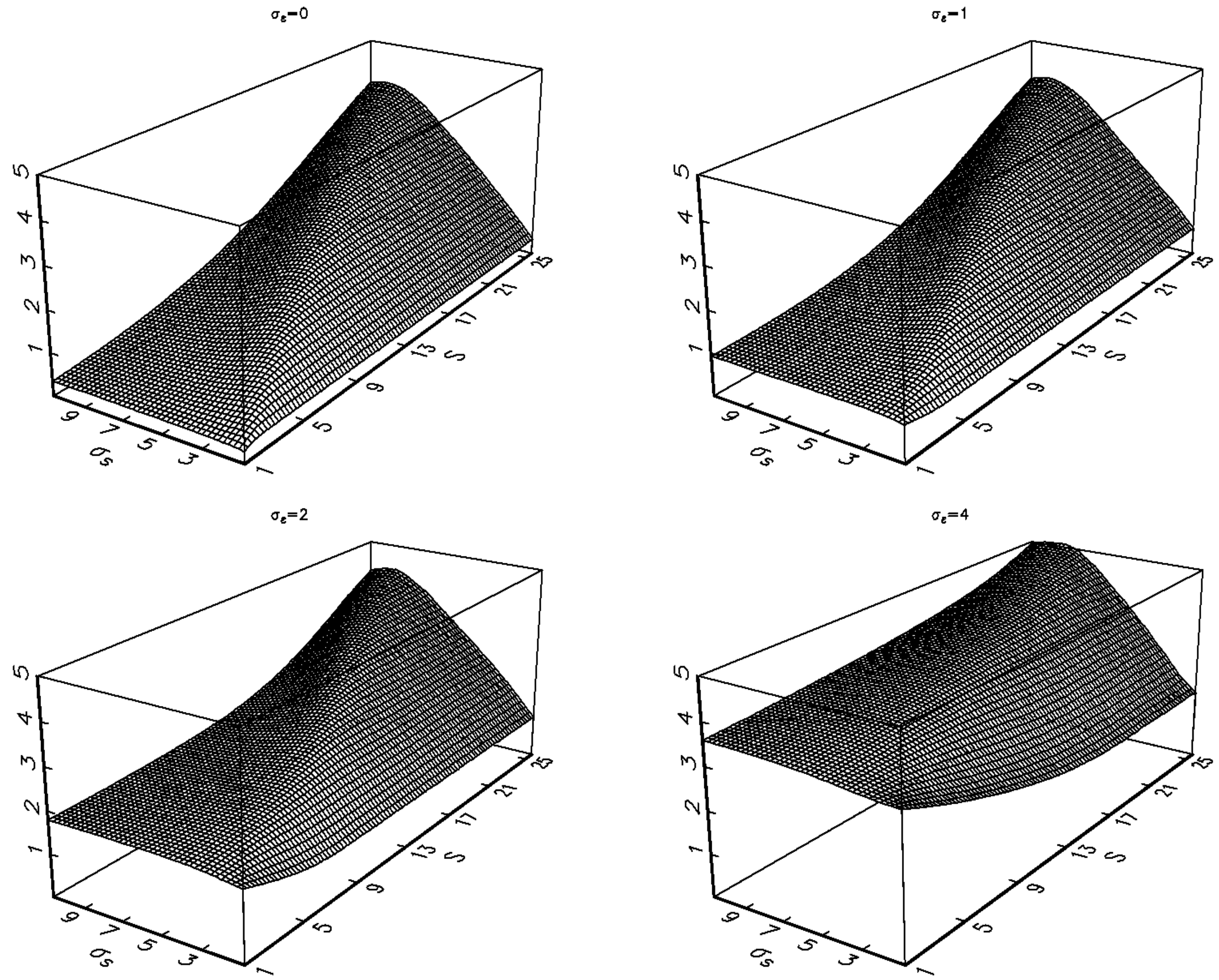
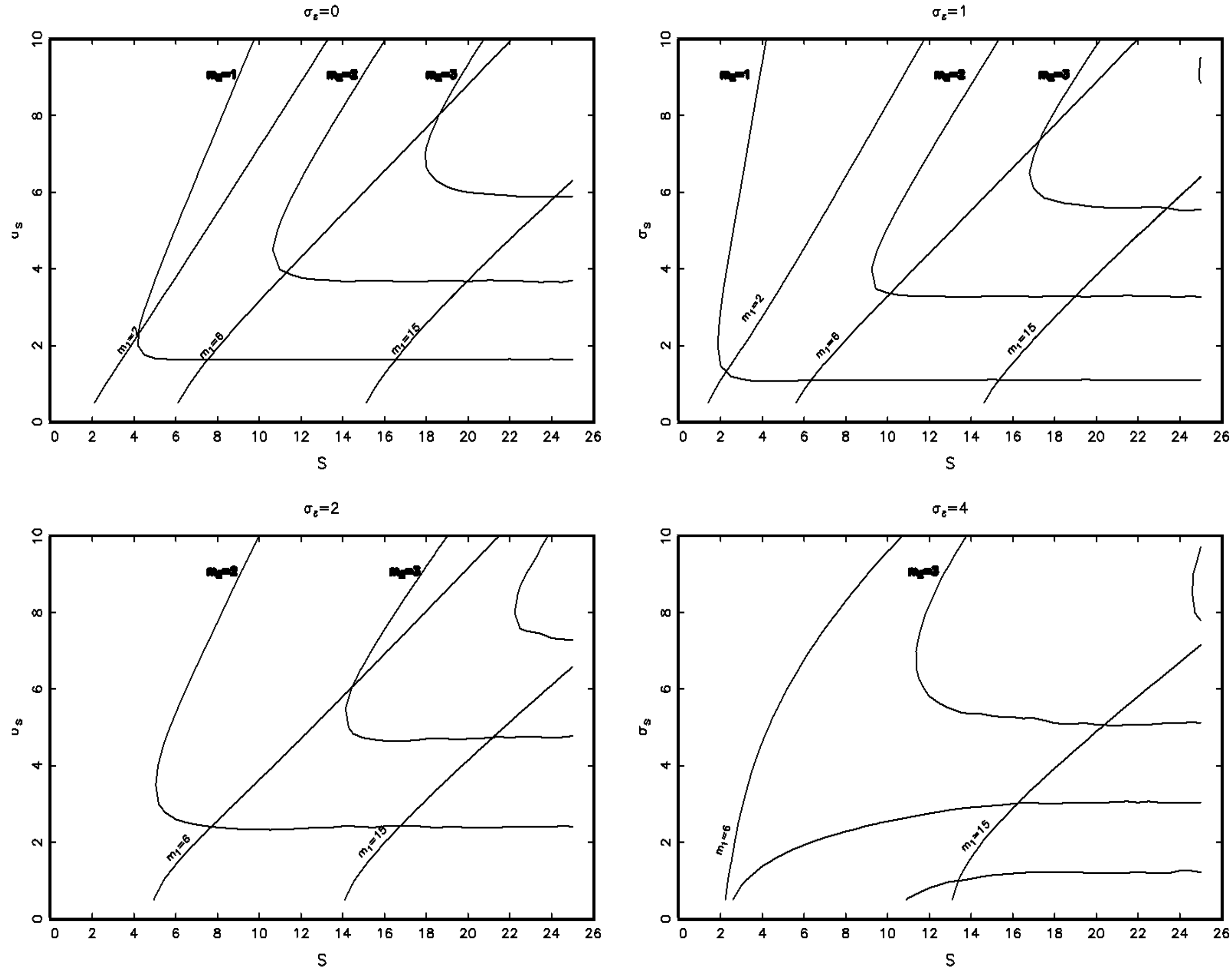


Figure D: Contour plots of mean and variance of price changes in the plane (S, σ_S) for different values of σ_ε



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230. G. Cette and M. de Jong, “The Rocky Ride of Break-even-inflation rates,” january 2009

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